Symmetric Rendezvous Search on the Line using Move Patterns with Different Lengths

Patchrawat "Patch" Uthaisombut
Department of Computer Science, University of Pittsburgh
utp@cs.pitt.edu

Abstract

The player-symmetric rendezvous search problem on the line is considered. We introduce a new way to define mixed (randomized) strategies, and formalize a general scheme to compute the expected rendezvous time of mixed strategies. We introduce a strategy that has the following properties: (1) move patterns have different lengths, and (2) the probability of choosing a move pattern in the current round depends on move pattern used in the previous round. The expected rendezvous time of our strategy is 4.39306. We also introduce a scheme to compute lower bounds for the expected rendezvous value under a certain assumption. We use the scheme to obtain a lower bound of 3.95460. This is the first non-trivial lower bound for the problem.

1 Introduction

Rendezvous search problem is the problem of how players, randomly placed in a search region, can minimize the time required to meet. The rendezvous search problem models several important problems in real life. For example, suppose a group of hikers become separated in a snow storm. Without any means to send signals, what search strategy the hikers should follow to reunite? In a second example, when animal species with very spare population attempt to find a mate, finding a mate is a non-trivial task, and can be thought of as a rendezvous problem.

A rendezvous problem needs not be spatial. Suppose a submarine got damaged, and the crew become trapped in two rooms. To save themselves they have to coordinate their efforts, but first they have to make contact with each other. There are n wires that connect the two rooms. At discrete times, the crew in each room connect a telephone to a wire and say "hello". They wish to minimize the time when they first pick the same wire.

A rendezvous problem is *asymmetric* if the players can distinguish among themselves and possibly choose distinct strategies. For example, if a mother and a child become separated, a simple joint strategy is for the child to stay put, and for the mother to do exhaustive search. In the *symmetric* version, the players cannot distinguish the identity among themselves and need to use the same strategy. For example, suppose each hiker mentioned above carries a survival guide book that tells what to do when hikers get separated. What single advise on rendezvous strategy should be put in the book?

1.1 Problem Description

In the *player-symmetric rendezvous search problem on the line*, two players are placed randomly on the line metric space where the initial distance between them is a random variable drawn from a probability distribution F where μ is the mean initial distance and D is the maximum initial distance. The players know F. Initially, the players independently face either East or West, with all four cases equiprobable. The line has no landmarks and the players cannot tell which direction is East or West. The players do not know which direction the other player is. The players can move at the maximum speed of 1. The players only see one another when they occupy the same point. We say that a *rendezvous* occurs when this happens, and the time it takes for rendezvous to occur is called *rendezvous time*. A *strategy* instructs the player how to move. We consider the symmetric version of the problem, where the players must employ the same strategy. For any pure (deterministic) strategy, there is a 50% chance that rendezvous will not occur. This happens when the players face the same direction initially. Thus, only mixed (randomized) strategy should be considered. The goal is to find a mixed strategy that has the smallest expected rendezvous time.

Many strategies instruct the players to always move at speed 1, and change direction only at times which are integer multiples of $D/2$. Thus, we define a *step* to be a motion in either direction at speed 1 for $D/2$ time units. We assume that at any time, the player is facing either East or West. We also use E and +1 to denote East and W and -1 to denote West. When the player takes a step, it is either *forward* or *backward* relative to the direction he is facing. We assume that when the player takes a step, either forward or backward, he never turns his face. A forward step is represented by F, and B for a backward step. A *move pattern* is a sequence of steps. Equivalently, a move pattern is a string that can be generated by the regular expression $(F + B)^*$.

In this paper, we consider only strategies that always move at speed 1 and change direction only at integer multiples of $D/2$. To simplify the analysis, we only consider the case that the initial distance between the players is known exactly, that is, $\mu = D$ is a known number. Without loss of generality, we assume that $D = 2$ throughout the paper.

1.2 Previous Results

Alpern posed this problem in [1]. He proposed a mixed strategy that employs a 3-step move pattern. Each of the players randomly faces either direction equiprobably, and then uses the move pattern FBB. The players repeat the process until rendezvous occurs. The following simple calculation shows that the expected rendezvous time is 5. Let T be the expected time to rendezvous. With probability 1/4, the players face toward one another, and meet at time 1. With probability 1/4, they face away from one another, and meet at time 3. With probability 1/2, they face the same direction (either East or West), and after 3 time units their distance remains 2, and they are in the same situation as the beginning. Then $T = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 3 + \frac{1}{2} \cdot (3 + T)$, and thus $T = 5$. Alpern also mentioned a simple strategy that employs a 2-step move pattern FB. The players pick a random direction before following the move pattern. The expected rendezvous time for this strategy is 7.

Anderson and Essegaier [5] improved the bound to 4.56676 with four 6-step move patterns: $FFBBBB, FBBBFF, FBBFBB$, and $FBFBBB$. Baston [7] later improved the bound to 4.4182 with four 7-step move patterns: $FBBFFBB, FBBBFFF, FBFBBBF$, and $FFBBBBB$. He also considered the case where there are 3 players. Discrete rendezvous problems (like the "telephone problem") was introduced in [6]. Rendezvous search on the circle was considered in [8, 2]. For a survey in rendezvous theory, please refer to [3]. A detail treatment in rendezvous theory is given in the book by Alpern and Gal [4].

1.3 Summary of Our Contributions

Our contributions on player-symmetric rendezvous search on the line are the following:

1. We introduce a new way to describe mixed (randomized) strategies.
2. We formalize a general technique for computing expected rendezvous time.
3. Based on 1 and 2, we improved the upper bound by giving a mixed strategy that uses (a) move patterns of different lengths, and (b) the probability of choosing a move pattern in the current round depends on move pattern used in the previous round.
4. Based on 1 and 2, we introduce a scheme to compute general lower bounds among strategies that consist of unit steps. This is the first non-trivial lower bound for this problem.

The rest of the paper is organized as follows. Section 2 discusses common properties of existing rendezvous strategies, a new way to describe mixed strategies, and two mixed strategies. In section 3 we describe a formal method in computing expected rendezvous time, compute the expected rendezvous time of one of our mixed strategies, and describe a scheme to compute general lower bounds. The paper finished with section 4 that lists some open problems.

2 Rendezvous Strategies

In section 2.1, we discuss about common properties of existing rendezvous strategies and how these properties lead to a simple equality to compute their expected rendezvous time. In section 2.2, we describe a new way to describe mixed strategies. In section 2.3, we describe 2 new strategies.

2.1 Properties of Strategies in the Literature

All strategies in the literature [1, 5, 7] for the symmetric rendezvous search on the line share the properties listed below.

1. A move pattern is a sequence of steps where a step is a motion in either direction for 1 time unit at speed 1.
2. A strategy consists of a finite (and small) set of move patterns that are used over and over.
3. All move patterns have the same length.
4. At any time, if the rendezvous has not occurred, the distance between the players is bounded by a constant.
5. At the end of each round, if the rendezvous has not occurred, the distance between the players is 2.
6. A move pattern is used with equal probability in either direction.
7. The probability of a using move pattern does not depend on the history of moves made so far.

Next, we the discuss how to compute the expected rendezvous time of strategies that satisfy all these properties. Suppose Γ is the set of move patterns and γ_i's are the elements of Γ. Suppose each of the move patterns has length L. Suppose p_i is the probability that move pattern γ_i is picked. Let $R(\sigma, \psi, d_1, d_2)$ be the rendezvous time of the players when the initial distance is 2, player 1 uses move pattern σ facing direction d_1, and player 2 uses move pattern ψ facing direction d_2. If the rendezvous does not occur, the value is undefined. Let T denotes the expected rendezvous time of this strategy. Then we have the following equality.

$$T = \frac{1}{4} \sum_{d_1,d_2=\pm 1} \sum_{i,j:\gamma_i,\gamma_j \in \Gamma} p_i p_j t_{i,j}^{d_1,d_2} \qquad \text{where} \tag{1}$$

$$t_{i,j}^{d_1,d_2} = \begin{cases} R(\gamma_i, \gamma_j, d_1, d_2) & \text{if rendezvous occurs using } \gamma_i \text{ and } \gamma_j \\ L + T & \text{if rendezvous does not occur using } \gamma_i \text{ and } \gamma_j \end{cases}$$

Note that if $t_{i,j}^{d_1,d_2} = L + T$, this mean that after the players spend L time units in executing move patterns γ_i and γ_j, they do not meet, and they end up in the same situation as when they start (being 2 units apart). Note that p_i is given and $R(\gamma_i, \gamma_j, d_1, d_2)$ can be computed in a straight forward fashion. After simplifying the right side of equality above, we have the equality $T = aT + b$ where a and b are scalars. An example computation for strategy FBB is given in section 1.2.

2.2 An Alternate Way to Describe a Mixed Strategy

In this section, we first briefly discuss about formal definition of pure and mixed strategies in the literature. Then we describe a new way to describe mixed strategies. In the literature, a *pure (deterministic) strategy* is a continuous path with maximum speed 1 that starts at position 0, that is, it can be defined as a function f satisfying

$$f : R^+ \to R \ \text{ such that } \ f(0) = 0 \ \text{ and } \ |f(t) - f(t')| \leq |t - t'|$$

where R^+ is the set of non-negative real numbers and R is the set of real numbers. A player who employs strategy f starting at position x and facing direction d where $d = \pm 1$ will be at position $x + d \cdot f(t)$ at time t. In the literature, a *mixed (randomized) strategy* is described as a probability distribution over pure strategies. While these definitions are simple and intuitive, they cannot be used directly to compute the expected rendezvous time of a mixed strategy. We will give an alternate way to describe mixed strategies.

Consider some point in time during the search game. We define the *complete move history* of a player to be the sequence of all moves the player has made so far. It can be described by a sequence of symbols that begins with S followed by 0 or more F's and B's where S represents the start of the search game, and F's and B's represent forward and backward steps relative to the direction of the player at the start of the search game. Equivalently, a complete move history is a string that can be generated by the regular expression $S(F+B)^*$.

A *mixed strategy* is described by specifying, for each possible complete move history σ, the probability of going 1 step forward, denoted by $P(F \mid \sigma)$, or going 1 step backward, denoted by $P(F \mid \sigma)$. Note that $P(F \mid \sigma) + P(B \mid \sigma) = 1$.

Strategies in the literature and our strategies use a set of multi-step move patterns over and over. Alternatively, a *complete move history* can be thought of as a string that begins with S followed by a sequence of move patterns. Suppose $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ is the set of move patterns used by a mixed strategy. Thus, a complete move history is a string that can be generated by the regular expression $S\,\Gamma^*$. For example, $S\gamma_4\gamma_2\gamma_2\gamma_3$ means the player has moved 4 rounds since the start of the search game by following move pattern γ_4 once, γ_2 twice, and γ_3 once in that order. A *mixed strategy* is described by specifying, for each possible complete move history σ, the probability of using a move pattern in Γ. This is denoted by $P(\gamma_i \mid \sigma)$. Note that for any σ, $\sum_{\gamma_i \in \Gamma} P(\gamma_i \mid \sigma) = 1$.

It is common for a strategy to use a move pattern α in opposite direction. We call this move pattern the *negation* of α, Specifically, the negation of a move pattern α, denoted $-\alpha$, is obtained from α by replacing all F's with B's and replacing all B's with F's. For example, if $\sigma = FBB$, then $-\sigma = BFF$. The concept also extends to move history. For example, if $\sigma = SFBB$, then $-\sigma = SBFF$. We sometimes write $d\alpha$ where $d = \pm 1$. If $d = +1$, then $d\alpha = \alpha$. If $d = -1$, then $d\alpha = -\alpha$. Note also that executing $-\alpha$ while facing direction d is the same as executing α while facing direction $-d$.

We give some examples of usage of notations defined in this section. Alpern's strategy [1] uses the patterns in the set $\Gamma = \{FBB, BFF\}$. Note that the the second move pattern is the negation of the first. The probability is given by $P(FBB \mid \sigma) = P(BFF \mid \sigma) = 1/2$ where σ is any history generated by $S\,\Gamma^*$. Property 6 in the previous section can be stated as $P(\alpha \mid \sigma) = P(-\alpha \mid \sigma)$ for any move pattern $\alpha \in \Gamma$ and any history σ generated by $S\,\Gamma^*$. Property 7 can be stated as $P(\alpha \mid \sigma_1) = P(\alpha \mid \sigma_2)$ for any move pattern $\alpha \in \Gamma$ and any histories σ_1 and σ_2 generated by $S\,\Gamma^*$.

2.3 Overview of Our strategies

In this section, we describe two mixed strategies for the symmetric rendezvous problem on the line. Both strategies use move patterns of different lengths. Thus, both strategies violate property 3 in section 2.1 and thus renders property 5 inapplicable. The second strategy also uses history-dependent probability distribution, thus, violating properties 6 and 7. Their expected rendezvous times are 4.40246 and 4.39306, respectively.

Move patterns of different lengths

A carefully crafted set of move patterns that are long can handle many contingency for rendezvous to occur; rendezvous occurs in many cases in the first round. This makes the expected rendezvous time small. However, there is some chance that the players choose the same move pattern and the same direction. When this happens, the time spent for that round is wasted. The longer the move pattern, the more time wasted. These two forces work in opposite direction. Our idea is to try to get the best of both worlds by using move patterns of different lengths.

However, this presents some difficulties. Obviously, the players could finish each move at different times. Thus property 5 in the section 2.1 no longer makes sense. Without this property, it is unclear how to determine the expected rendezvous time. The next best thing one can hope for is that the strategy still satisfies property 4. Then an set of equalities can be constructed to determine the expected rendezvous time. We will discuss a technique for computing this in section 3. Our first strategy is given in figure 2.3. The first four move patterns have length 7. The last one has length 6. The strategy satisfies property 4. This strategy yields an expected rendezvous time of **4.40246**. We omit the analysis for this strategy because of space limitation and because our second strategy performs better.

Mixed Strategy 1

Γ is the set of the following 5 move patterns γ_i's as well as their negations.

$\gamma_1 = FFBBBBB$, $\gamma_2 = FBFBBBF$, $\gamma_3 = FBBFBFB$, $\gamma_4 = FBBFBBF$, $\gamma_5 = FBBBFF$.
$p_1 = 0.245521$, $p_2 = 0.184934$, $p_3 = 0.174242$, $p_4 = 0.0709467$, $p_5 = 0.324357$.

$$P(\gamma_i \mid \sigma) = P(-\gamma_i \mid \sigma) = \frac{1}{2} p_i \quad \text{where } \sigma \text{ is any history generated by } S\,\Gamma^*.$$

Figure 1: Mixed Strategy 1

History-dependent probability distribution

Strategies in the literature satisfies property 5 in section 2.1, that is, at the end of each round, if the rendezvous has not occurred, the distance between the players is 2. This is the same situation as the one at the start of the game. Therefore, it makes sense to use the same (best) probability distribution in every round independent of the moves made so far.

Consider our first strategy, which uses move patterns of different lengths. Obviously, the players could finish each round at different time, a different situation from the one at the start of the game. Thus, it is conceivable that the players can do better if they use different probability distributions for different situations. Our second strategy, given in figure 2.3, uses the same move patterns as in the first strategy. Its expected rendezvous time is **4.39306**. The analysis is given in section 3.3.

Mixed Strategy 2

Γ is the set of the following 5 move patterns γ_i's as well as their negations.
$\gamma_1 = FFBBBBB$, $\gamma_2 = FBFBBBF$, $\gamma_3 = FBBFBFB$, $\gamma_4 = FBBFBBF$, $\gamma_5 = FBBBFF$.

$$\begin{aligned}
P^F = P^B = [p_i] &= [\ 0.122875\ \ 0.089797\ \ 0.087084\ \ 0.038993\ \ 0.161250\ ]^T \\
Q^F = [q_i^F] &= [\ 0.004312\ \ 0.020515\ \ 0.024417\ \ 0.115811\ \ 0.188155\ ]^T \\
Q^B = [q_i^B] &= [\ 0.200483\ \ 0.153507\ \ 0.147767\ \ 0.000000\ \ 0.145028\ ]^T \\
R^F = [r_i^F] &= [\ 0.145684\ \ 0.108876\ \ 0.105318\ \ 0.041854\ \ 0.176428\ ]^T \\
R^B = [r_i^B] &= [\ 0.050613\ \ 0.069621\ \ 0.076064\ \ 0.058674\ \ 0.166861\ ]^T
\end{aligned}$$

$$\text{For all } j \qquad P(\gamma_i \mid S) = P(-\gamma_i \mid S) = p_i.$$

Suppose σ is any history generated by $S\,\Gamma^*\,(d\gamma_j)$ for some j and $d = \pm 1$, that is, σ is a sequence of move patterns in Γ ending with move pattern γ_j (if $d = +1$) or its negation (if $d = -1$).

$$\begin{aligned}
&\text{If } j = 1, 2, 4, \text{ then} && P(+d\gamma_i \mid \sigma) = P(-d\gamma_i \mid \sigma) = p_i \\
&\text{If } j = 3, \text{ then} && P(+d\gamma_i \mid \sigma) = q_i^F \quad \text{and} \quad P(-d\gamma_i \mid \sigma) = q_i^B \\
&\text{If } j = 5, \text{ then} && P(+d\gamma_i \mid \sigma) = r_i^F \quad \text{and} \quad P(-d\gamma_i \mid \sigma) = r_i^B
\end{aligned}$$

Figure 2: Mixed Strategy 2

For example, suppose the player use $-\gamma_5$ in the previous round, then the probability that he will use $+\gamma_2$ in the next round is equal to $P(\gamma_2 \mid S\,\Gamma^*\,(-\gamma_5)) = r_2^B = 0.069621$.

3 Computation

In the literature, a scheme equivalent to equation (1) is use to compute expected rendezvous time of strategies [1, 5, 7]. In section 3.1 and 3.2, we generalize this equation so that it applies to mixed strategies that use move patterns of different lengths and history-dependent probability distributions. In section 3.3, we use this equation to compute the expected rendezvous time of our second strategy. In section 3.4, we introduce a scheme to compute general lower bounds.

3.1 Notations and Computation

Suppose σ and ψ are move patterns, and i and j are integers.

- $\sigma[i]$ is the i'th symbol in σ.
- $\sigma[i..j]$ is the sequence of symbols in σ from the i'th to the j'th inclusive.
- $|\sigma|$ is the number of symbols in σ.
- $\|\sigma\|$ is net position of a player who executes σ facing East. This is equal to $\sum_{i=1}^{|\sigma|} \sigma[i]$. Here F is $+1$ and B is -1.
- Suppose initially both players face East and are x units apart. Suppose player 1 executes σ and player 2 executes ψ where $|\sigma| = |\psi|$. Then the distance between the players in the end, denoted $d(\alpha_1, \alpha_2, x)$, is equal to $x - \|\alpha_1\| + \|\alpha_2\|$.

Suppose initially both players face East and are x units apart. Suppose players 1 and 2 execute move patterns σ and ψ, respectively. The *rendezvous indicator* $I(\sigma, \psi, x)$ is 1 if the players meet by the end of their moves, and 0 otherwise, that is,

$$I(\sigma, \psi, x) = \begin{cases} 1 & \text{if there is } t \text{ such that } 0 \le t \le \min\{|\sigma|, |\psi|\} \text{ and } d(\sigma[1..t], \psi[1..t], x) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The *rendezvous time* is the earliest time that the players meet one another, and is given by

$$T(\sigma, \psi, x) = \min\{t : d(\sigma[1..t], \psi[1..t], x) = 0\}. \tag{2}$$

By convention, if the player don't meet, the rendezvous time is ∞.

To define the expected rendezvous time for players employing mixed strategies, consider the following situation. Our formulation allow the players to use different mixed strategies. Players 1 and 2 employ mixed strategies s_1^* and s_2^*, respectively. Let σ' and ψ' be some complete move histories where $|\sigma'| = |\psi'|$. Since the start of the search game, players 1 and 2 have moved according to σ' and ψ' in directions d_1 and d_2, respectively. Rendezvous has not occurred, and their distance at *present* is x. Players 1 and 2 are about to execute move patterns σ and ψ, respectively. Note that if rendezvous still does not occur after σ and ψ, then the players have to randomly choose more moves according to strategies s_1^* and s_2^* and histories σ' and ψ' until they meet one another. The expected amount of time from the *present* until rendezvous occurs is denoted by

$$T^*(s_1^*, s_2^*, d_1, d_2, x \mid \sigma', \psi';\, \sigma, \psi).$$

We now describe some equalities involving this quantity. There are two cases whether the players meet by the end of σ and ψ. If they meet, the time required is $T(d_1\sigma, d_2\psi, x)$.

If they do not meet by the end of σ and ψ, they have to move further according to s_1^*, s_2^*, σ', and ψ'. Let Γ_1 be the set of move patterns that player 1 could take assuming that he has taken $\sigma'\sigma$. Let Γ_2 be defined similarly. Suppose player 1 chooses α and player 2 chooses β. The probability that player 1 chooses $\alpha \in \Gamma_1$ after $\sigma'\sigma$ is $P_{s_1^*}(\alpha \mid \sigma'\sigma)$. Similarly, the probability that player 2 uses

$\beta \in \Gamma_2$ after $\psi'\psi$ is $P_{s_2^*}(\beta \mid \psi'\psi)$. The expected meeting time given that the players plan to use $\sigma\alpha$ and $\psi\beta$ is, by definition, $T^*(s_1^*, s_2^*, d_1, d_2, x \mid \sigma', \psi';\ \sigma\alpha, \psi\beta)$. Thus, if rendezvous does not occur by the end of σ and ψ, the expected meeting time is

$$\sum_{\alpha\in\Gamma_1}\sum_{\beta\in\Gamma_2} P_{s_1^*}(\alpha \mid \sigma_1)P_{s_2^*}(\beta \mid \sigma_2)T^*(s_1^*, s_2^*, d_1, d_2, x \mid \sigma', \psi';\ \sigma\alpha, \psi\beta)$$

The sum can be written as a product of matrices, $P_1^T\ T\ P_2$. Suppose the elements of Γ_1 and Γ_2 are ordered in some way, ie. $\Gamma_1 = \{\alpha_i \mid i = 1, ..., k\}$ and $\Gamma_2 = \{\beta_j \mid j = 1, ..., l\}$. Column matrices P_1 and P_2 and square matrix T is given as follows.

$$\begin{aligned} P_1 &= [p_i]_{k\times 1} \text{ where } p_i = P_{s_1^*}(\alpha_i \mid \sigma'\sigma), i = 1, ..., k \\ P_2 &= [p'_j]_{l\times 1} \text{ where } p'_j = P_{s_2^*}(\beta_j \mid \psi'\psi), j = 1, ..., l \\ T &= [t_{i,j}]_{k\times l} \text{ where } t_{i,j} = T^*(s_1^*, s_2^*, d_1, d_2, x \mid \sigma', \psi';\ \sigma\alpha_i, \psi\beta_j), i = 1, ..., k \text{ and } j = 1, ..., l. \end{aligned} \tag{3}$$

Therefore, from the development above,

$$T^*(s_1^*, s_2^*, d_1, d_2, x \mid \sigma', \psi';\ \sigma, \psi) = \begin{cases} T(d_1\sigma, d_2\psi, x) & \text{if } I(\sigma, \psi, x) = 1 \\ P_1^T T P_2 & \text{if } I(\sigma, \psi, x) = 0 \end{cases} \tag{4}$$

The *expected rendezvous time* with mixed strategies s_1^* and s_2^*, with initial distance x between the players, and with each of the players starts in either direction equiprobably, is given by

$$T^*(s_1^*, s_2^*, x) = \frac{1}{4}\sum_{d_1,d_2=\pm 1} T^*(s_1^*, s_2^*, d_1, d_2, x \mid S, S;\ \varepsilon, \varepsilon) \tag{5}$$

where ε represents a move pattern of length 0.

The *symmetric rendezvous value* when the initial distance is 2 is defined as the infimum of the expected meeting time, over all mixed strategies, and is given by

$$R^s = \inf_{s^*}\ T^*(s^*, s^*, 2).$$

Note, when a player executes a move pattern, he does not know whether he is facing East or West. However, in the definition for $T^*(...)$ in (4), the directions are explicitly indicated using d_1 and d_2. For $I(\sigma, \psi, x)$ and $T(\sigma, \psi, x)$, the directions are implicit within σ and ψ. If player 1 executes α in direction +1, then $\sigma = +\alpha$. If player 1 executes α in direction -1, then $\sigma = -\alpha$.

3.2 Decomposition

In this section, we decompose the computation of $I(\sigma, \psi, x)$, $T(\sigma, \psi, x)$, and $T^*(...)$ on long move patterns into computations on shorter move patterns. Let σ and ψ be any move patterns. Suppose σ_1, σ_2, ψ_1, and ψ_2 are move patterns such that
(a) $\sigma_1\sigma_2 = \sigma$ and $\psi_1\psi_2 = \psi$
(b) $|\sigma_1| = |\psi_1|$, and
(c) $I(\sigma_1, \psi_1, x) = 0$.
Then the following holds.

$$I(\sigma, \psi, x) = I(\sigma_2, \psi_2, d(\sigma_1, \psi_1, x)) \tag{6}$$

$$T(\sigma, \psi, x) = |\sigma_1| + T(\sigma_2, \psi_2, d(\sigma_1, \psi_1, x)) \tag{7}$$

$$T^*(s_1^*, s_2^*, d_1, d_2, x \mid \sigma', \psi';\ \sigma, \psi) = |\sigma_1| + T^*(s_1^*, s_2^*, d_1, d_2, d(d_1\sigma_1, d_2\psi_1, x) \mid \sigma'\sigma_1, \psi'\psi_1;\ \sigma_2, \psi_2) \tag{8}$$

We briefly argue why these equalities hold. Consider the situation where the players execute the first $|\sigma_1|$ steps of σ and ψ. This is exactly σ_1 and ψ_1. From the assumption, rendezvous does not occur. At this point, the distance between the players is $d(\sigma_1, \psi_1, x)$ (or $d(d_1\sigma_1, d_2\psi_1, x)$ in case of T^*), and the remaining moves are given by σ_2 and ψ_2. From this point on, the players meet if and only if $I(\sigma_2, \psi_2, d(\sigma_1, \psi_1, x)) = 1$, and if they do, the rendezvous time from this point is $T(\sigma_2, \psi_2, d(\sigma_1, \psi_1, x))$. For T^*, the complete move history at this point is $\sigma'\sigma_1$ and $\psi'\psi_1$. The expected rendezvous time from this point on is given by $T^*(s_1^*, s_2^*, d_1, d_2, d(d_1\sigma_1, d_2\psi_1, x) \mid \sigma'\sigma_1, \psi'\psi_1;\, \sigma_2, \psi_2)$.

3.3 Computation of Upper Bound

In this section, we give the detail for computing the expected rendezvous time of our second mixed strategy. First, we give an addition definition. If σ is a *complete move history*, then a *recent move history* is any suffix of σ. In each round of our strategy, a move pattern and a direction is randomly chosen based only on the direction and the move pattern of the *previous* round. For such strategies, definition of T^* based on *recent* move histories can be adapted in a natural way from the definition of T^* based on *complete* move histories. Let s^* denote our second mixed strategy given in figure 2.3. If σ and ψ are generated from $S\,\Gamma^*(d_3\gamma_i)$ and $S\,\Gamma^*(d_4\gamma_j)$, respectively, where $d_3, d_4 = \pm 1$, then

$$T^*(s^*, s^*, d_1, d_2, 2 \mid \sigma, \psi;\, \sigma', \psi') = T^*(s^*, s^*, d_1, d_2, 2 \mid d_3\gamma_i, d_4\gamma_j;\, \sigma', \psi').$$

We define a few quantities that are needed in computing the expected rendezvous time of s^*.

$$\begin{aligned} T &= T^*(s^*, s^*, +1, +1, 2 \mid S, S;\, \varepsilon, \varepsilon) & T_{5,5} &= T^*(s^*, s^*, +1, +1, 2 \mid \gamma_5, \gamma_5; \varepsilon, \varepsilon) \\ T_{3,3} &= T^*(s^*, s^*, +1, +1, 2 \mid \gamma_3, \gamma_3; \varepsilon, \varepsilon) & T_{5,3} &= T^*(s^*, s^*, +1, +1, 2 \mid \gamma_5, \gamma_3[1..6];\, \varepsilon, \gamma_3[7]) \\ T_{4,3} &= T^*(s^*, s^*, +1, +1, 2 \mid \gamma_4, \gamma_3; \varepsilon, \varepsilon) & &= T^*(s^*, s^*, +1, +1, 2 \mid FBBBFF, FBBFBF;\, \varepsilon, B) \end{aligned}$$

If the recent move history of players 1 and 2 was γ_i and γ_j, respectively, where $(i, j) =$ (1,1), (2,2), (4,2), and (4,4), then to choose the next move, the players use the same probability distribution as the case when the move history for both is S (no move has been made). Thus, for these (i, j)'s,

$$T^*(s^*, s^*, +1, +1, 2 \mid \gamma_i, \gamma_j;\, \varepsilon, \varepsilon) = T^*(s^*, s^*, +1, +1, 2 \mid S, S;\, \varepsilon, \varepsilon) = T.$$

Since $P(-\gamma_i \mid S) = P(\gamma_i \mid S)$, and the probability of choosing a move pattern γ_i (or $-\gamma_i$) depends only on the direction and the move pattern in the previous move (and not on the direction of the player at the start of the game), then

$$T^*(s^*, s^*, d_1, d_2, 2 \mid S, S;\, \varepsilon, \varepsilon) = T^*(s^*, s^*, +1, +1, 2 \mid S, S;\, \varepsilon, \varepsilon) \quad \text{for } d_1, d_2 = \pm 1. \tag{9}$$

From (5) and (9), the expected rendezvous time of our mixed strategy is

$$\begin{aligned} T^*(s^*, s^*, 2) &= \frac{1}{4} \sum_{d_1, d_2 = \pm 1} T^*(s^*, s^*, d_1, d_2, 2 \mid S, S;\, \varepsilon, \varepsilon) = T^*(s^*, s^*, +1, +1, 2 \mid S, S;\, \varepsilon, \varepsilon) \\ &= T = \begin{bmatrix} P^F \\ P^B \end{bmatrix}^T \begin{bmatrix} A^{+1,+1} & A^{+1,-1} \\ A^{-1,+1} & A^{-1,-1} \end{bmatrix} \begin{bmatrix} P^F \\ P^B \end{bmatrix} \quad \text{from (4)} \end{aligned} \tag{10}$$

where P is defined in the figure 2.3 and A is given below.

$$\begin{aligned} A^{+1,+1} &= \begin{bmatrix} 7+T & 2 & 2 & 2 & 2 \\ 7 & 7+T & 3 & 3 & 3 \\ 6 & 6 & 7+T_{3,3} & 6 & 4 \\ 7 & 7+T & 7+T_{4,3} & 7+T & 4 \\ 6 & 6 & 6+T_{5,3} & 6 & T_{5,5} \end{bmatrix} \quad A^{-1,+1} = \begin{bmatrix} 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 3 & 3 & 3 \\ 5 & 5 & 3 & 3 & 3 \\ 4 & 4 & 3 & 3 & 3 \end{bmatrix} \\ A^{-1,-1} &= (A^{+1,+1})^T \qquad A^{+1,-1} = [1]_{5\times 5} \end{aligned} \tag{11}$$

For $i, j = 1, ..., 5$ and $d_1, d_2 = \pm 1$, entry (i, j) in A^{d_1,d_2} is the expected rendezvous time when player 1 uses move pattern γ_i in direction d_1 and player 2 uses γ_j in direction d_2. This is given by $T^*(s^*, s^*, d_1, d_2, x \mid S, S;\ \gamma_i, \gamma_j)$. Entries with numerical values are computed using equality (2). All entries containing T or $T_{i,j}$ are computed using equality (8). All such entries except entry $(5, 3)$ are computed as follows.

$$\begin{aligned} T^*(s^*, s^*, +1, +1, 2 \mid S, S;\ \gamma_i, \gamma_j) &= |\gamma_i| + T^*(s^*, s^*, +1, +1, d(+\gamma_i, +\gamma_j, 2) \mid S\gamma_i, S\gamma_j;\ \varepsilon, \varepsilon) \\ &= |\gamma_i| + T^*(s^*, s^*, +1, +1, 2 \mid \gamma_i, \gamma_j;\ \varepsilon, \varepsilon) \\ &= \begin{cases} 7 + T & \text{for } (i,j) = (1,1),\ (2,2),\ (4,2), \text{ and } (4,4) \\ 7 + T_{3,3} & \text{for } (i,j) = (3,3) \\ 7 + T_{4,3} & \text{for } (i,j) = (4,3) \\ 6 + T_{5,5} & \text{for } (i,j) = (5,5) \end{cases} \end{aligned}$$

Entry (5,3), shown below, is computed differently because γ_5 and γ_3 have different lengths.

$$\begin{aligned} & T^*(s^*, s^*, +1, +1, 2 \mid S, S;\ \gamma_5, \gamma_3) \\ =\ & |\gamma_5| + T^*(s^*, s^*, +1, +1, d(\gamma_5, \gamma_3[1..6], 2) \mid S\gamma_5, S\gamma_3[1..6];\ \varepsilon, \gamma_3[7]) \\ =\ & 6 + T^*(s^*, s^*, +1, +1, 2 \mid \gamma_5, \gamma_3[1..6];\ \varepsilon, \gamma_3[7]) \ = \ 6 + T_{5,3} \end{aligned}$$

Next give equalities on $T_{3,3}, T_{4,3}, T_{5,3}$, and $T_{5,5}$ based on (4).

$$\begin{aligned} T_{3,3} &= \begin{bmatrix} Q^F \\ Q^B \end{bmatrix}^T \begin{bmatrix} A^{+1,+1} & A^{+1,-1} \\ A^{-1,+1} & A^{-1,-1} \end{bmatrix} \begin{bmatrix} Q^F \\ Q^B \end{bmatrix} & T_{4,3} &= \begin{bmatrix} P^F \\ P^B \end{bmatrix}^T \begin{bmatrix} A^{+1,+1} & A^{+1,-1} \\ A^{-1,+1} & A^{-1,-1} \end{bmatrix} \begin{bmatrix} Q^F \\ Q^B \end{bmatrix} \\ T_{5,5} &= \begin{bmatrix} R^F \\ R^B \end{bmatrix}^T \begin{bmatrix} A^{+1,+1} & A^{+1,-1} \\ A^{-1,+1} & A^{-1,-1} \end{bmatrix} \begin{bmatrix} R^F \\ R^B \end{bmatrix} & T_{5,3} &= \begin{bmatrix} R^F \\ R^B \end{bmatrix}^T \begin{bmatrix} B^{+1,+1} & B^{+1,-1} \\ B^{-1,+1} & B^{-1,-1} \end{bmatrix} \begin{bmatrix} Q^F \\ Q^B \end{bmatrix} \end{aligned} \tag{12}$$

where P, Q, R are defined in figure 2.3, A is given in (11), and B is given below.

$$B^{+1,+1} = B^{+1,-1} = [1]_{5\times 5} \quad B^{-1,+1} = \begin{bmatrix} 5 & 5 & 4 & 4 & 4 \\ 5 & 5 & 4 & 4 & 4 \\ 5 & 3 & 3 & 3 & 3 \\ 5 & 3 & 3 & 3 & 3 \\ 4 & 3 & 3 & 3 & 3 \end{bmatrix} \quad B^{-1,-1} = \begin{bmatrix} 3 & 4 & 5 & 5 & 6 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

By evaluating equations in (10) and (12), we obtain 5 equalities in 5 variables. Solving the equalities yields $T = 4.39306$.

3.4 Computation of General Lower Bound

This in section we describe a scheme to compute general lower bounds among the restricted class of strategies that move in steps of length $D/2$. The lower bound is based on the simple idea that if the players locate x units apart, then the lower bound for the expected rendezvous time is $x/2$, which is the time for the players to meet if they move toward each other at speed 1. That is, for any move histories σ and ψ,

$$T^*(s^*, s^*, d_1, d_2, x \mid \sigma, \psi; \varepsilon, \varepsilon) \geq x/2. \tag{13}$$

We can use this idea to obtain a lower bound for the expected rendezvous time for any mixed strategy s^* by modifying it into a pseudo strategy r^* as follow. Let the players follows s^* for k time steps for some integer k. If rendezvous has not occurred, let the players move toward one another at speed 1. Clearly, the expected rendezvous time of r^* is a lower bound for that of s^*.

Recall that a mixed strategy is described by specifying $P(F \mid \sigma)$ for any complete move history σ generated by $S(F+B)^*$. We can translate this into a strategy that is defined based on move patterns of length k. For $k \geq 1$, let $\Gamma^{(k)}$ be the set of all move patterns of length k. For example, $\Gamma^{(2)} = \{FF, FB, BF, BB\}$. Note that $|\Gamma^{(k)}| = 2^k$. Suppose γ_i's are the elements of $\Gamma^{(k)}$. For any complete history σ generated by $S\,\Gamma^{(k)*}$, $P(\gamma_i \mid \sigma)$ is given by

$$P(\gamma_i \mid \sigma) = \Pi_{j=1}^{k} P(\gamma_i[j] \mid \sigma\gamma_i[1..j-1]).$$

From (4), the expected rendezvous time for any mixed strategy s^* is

$$T^*(s^*, s^*, 2) = \frac{1}{4} \sum_{d_1, d_2 = \pm 1} T^*(s^*, s^*, d_1, d_2, 2 \mid S, S;\, \varepsilon, \varepsilon) = \frac{1}{4} \sum_{d_1, d_2 = \pm 1} P^T A^{d_1, d_2} P$$

where $P = [p_i]$ is a $2^k \times 1$ matrix and $A^{d_1,d_2} = [a_{i,j}^{d_1,d_2}]$ is a $2^k \times 2^k$ matrix defined as follows.

$$\begin{aligned} p_i &= P_{s^*}(\gamma_i \mid S), \quad i = 1, ..., 2^k \\ a_{i,j}^{d_1,d_2} &= T^*(s^*, s^*, d_1, d_2, 2 \mid S, S; \gamma_i, \gamma_j) \\ &= \begin{cases} T(d_1\gamma_i, d_2\gamma_j, 2) & \text{if } I(d_1\gamma_i, d_2\gamma_j, 2) = 1 \\ k + T(s^*, s^*, d_1, d_2, d(d_1\gamma_i, d_2\gamma_j, 2) \mid S\gamma_i, S\gamma_j; \varepsilon, \varepsilon) & \text{if } I(d_1\gamma_i, d_2\gamma_j, 2) = 0 \end{cases} \\ &\quad \text{from (4) and (8)} \end{aligned}$$

Define $B^{d_1,d_2} = [b_{i,j}^{d_1,d_2}]$ for $d_1, d_2 = \pm 1$ to be a $2^k \times 2^k$ matrix where

$$b_{i,j}^{d_1,d_2} = \begin{cases} T(d_1\gamma_i, d_2\gamma_j, 2) & \text{if } I(d_1\gamma_i, d_2\gamma_j, 2) = 1 \\ k + d(d_1\gamma_i, d_2\gamma_j, 2)/2 & \text{if } I(d_1\gamma_i, d_2\gamma_j, 2) = 0 \end{cases}$$

Thus, from (13), $a_{i,j}^{d_1,d_2} \geq b_{i,j}^{d_1,d_2}$ for $d_1, d_2 = \pm 1$ and $i, j = 1, ..., 2^k$. Note that the value of $b_{i,j}^{d_1,d_2}$ can be computed in a straightforward fashion and depends only on $\Gamma^{(k)}$ and not on s^*. Thus,

$$R^s \;=\; \inf_{s^*} T^*(s^*, s^*, 2) \;=\; \inf_{s^*} \frac{1}{4} \sum_{d_1,d_2=\pm 1} P^T A^{d_1,d_2} P \;\geq\; \inf_{s^*} \frac{1}{4} \sum_{d_1,d_2=\pm 1} P^T B^{d_1,d_2} P$$

By finding a global minimum for $(1/4)\sum_{d_1,d_2=\pm 1} P^T B^{d_1,d_2} P$, we obtain a lower bound for R^s. Table 1 shows the computed general lower bounds for $k = 1, ..., 7$.

k	1	2	3	4	5	6	7
lower bound	2	2.5	3	55/16 > 3.43	175/48 > 3.64	1993/520 > 3.83	190327/48128 > **3.95460**

Table 1: Lower bounds.

4 Open Questions

Can the upper bound be improved by a strategy that considers a longer history or the complete move history? And by how much? How to compute the lower bound for $k \to \infty$? How close is this lower bound to R^s? How to adapt the computation technique to other rendezvous problems?

References

[1] Steve Alpern. The rendezvous search problem. *SIAM Journal on Control and Optimization*, 33(3):673–683, 1995.

[2] Steve Alpern. Asymmetric rendezvous search on the circle. *Dynam. Control*, 10:33–45, 2000.

[3] Steve Alpern. Rendezvous search: A personal perspective. *Operations Research*, 50(5):772–795, 2002.

[4] Steve Alpern and Shmuel Gal. *The Theory of Search Games and Rendezvous*. Kluwer Academic Publishers, 2003.

[5] Edward J. Anderson and Skander Essegaier. Rendezvous search on the line with indistinguishable players. *SIAM Journal on Control and Optimization*, 33(6):1637–1642, 1995.

[6] E.J. Anderson and R.R. Weber. The rendezvous problem on discrete locations. *J. Appl. Probab.*, 28:839–851, 1990.

[7] Vic Baston. Note: Two rendezvous search problems on the line. *Naval Research Logistics*, 46(3):335–340, 1999.

[8] J.V. Howard. Rendezvous search on the interval and circle. *Oper. Res.*, 47(4):550–558, 1999.