

**CS 441 Discrete Mathematics for CS
Lecture 9**

**Inverse functions and composition.
Sequences.**

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Course administration

Midterm:

- **Tuesday, October 6, 2009**
- **Closed book, in-class**
- **Covers Chapters 1 and 2.1-2.3 of the textbook**

Homework 4 is out

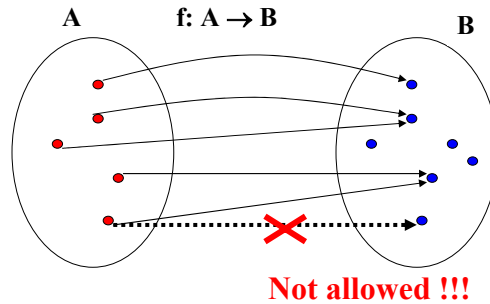
- **Due on Thursday, October 1, 2006**

Course web page:

<http://www.cs.pitt.edu/~milos/courses/cs441/>

Functions

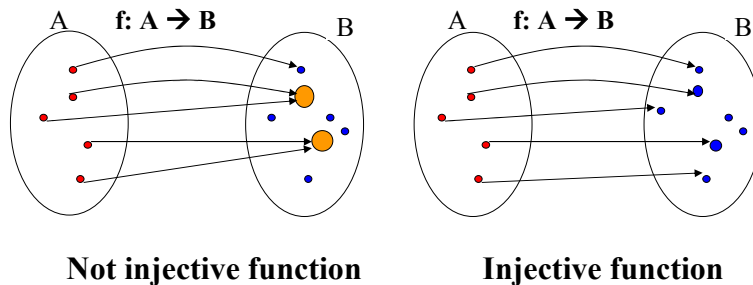
- **Definition:** Let A and B be two sets. A **function from A to B** , denoted $f: A \rightarrow B$, is an assignment of exactly one element of B to each element of A . We write $f(a) = b$ to denote the assignment of b to an element a of A by the function f .



Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f . A function is said to be an **injection if it is one-to-one**.

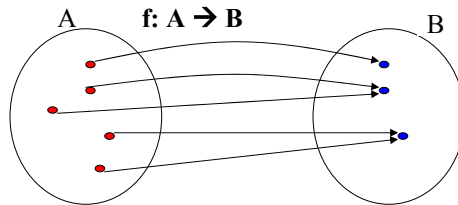
Alternate: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$. This is the contrapositive of the definition.



Surjective function

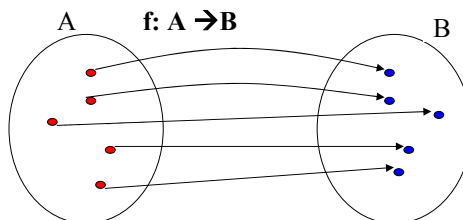
Definition: A function f from A to B is called **onto, or surjective**, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

Alternative: all co-domain elements are covered



Bijjective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.



Bijjective functions

- Let f be a function from a set A to itself, that is

$$f: A \rightarrow A$$

Assume

- **A is finite and f is one-to-one (injective)**
- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have $|A|$ different elements A points to. Since $f: A \rightarrow A$ the co-domain is covered thus the function is also a surjection (and bijection)
- **A is finite and f is an onto function**
- Is the function one-to-one?

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- **A is finite and f is an onto function**
- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

Functions on real numbers

Definition: Let f_1 and f_2 be functions from A to \mathbf{R} (reals). Then $f_1 + f_2$ and $f_1 * f_2$ are also functions from A to \mathbf{R} defined by

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 * f_2)(x) = f_1(x) * f_2(x)$.

Examples:

- **Assume**

- $f_1(x) = x - 1$
- $f_2(x) = x^3 + 1$

then

- $(f_1 + f_2)(x) = x^3 + x$
- $(f_1 * f_2)(x) = x^4 - x^3 + x - 1$.

Increasing and decreasing functions

Definition: A function f whose domain and codomain are subsets of real numbers is **strictly increasing** if $f(x) > f(y)$ whenever $x > y$ and x and y are in the domain of f . Similarly, f is called **strictly decreasing** if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f .

Note: Strictly increasing and strictly decreasing functions are one-to-one.

Example:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$. Is it increasing ?

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Note: Strictly increasing and strictly decreasing functions are one-to-one (injective).

Example:

• Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$. Is it increasing ?

• **Proof .**

For $x > y$ holds $2x > 2y$ and subsequently $2x - 1 > 2y - 1$

Thus g is strictly increasing.

Identity function

Definition: Let A be a set. The **identity function** on A is the function $i_A : A \rightarrow A$ where $i_A(x) = x$.

Example:

• Let $A = \{1, 2, 3\}$

Then:

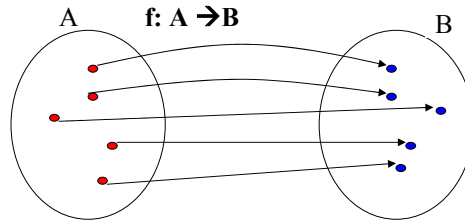
• $i_A(1) = 1$

• $i_A(2) = 2$

• $i_A(3) = 3$.

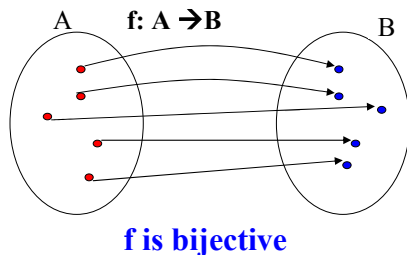
Bijjective functions

Definition: A function f is called a **bijection** if it is **both one-to-one and onto**.



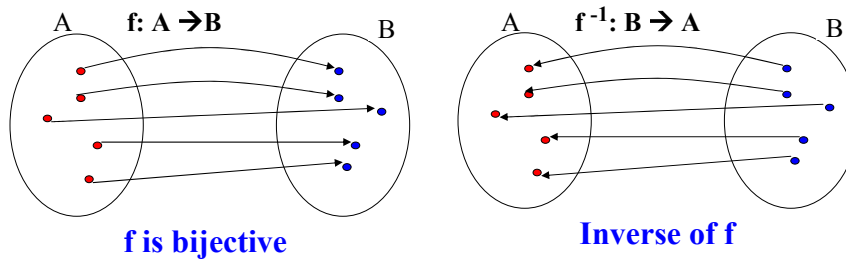
Inverse functions

Definition: Let f be a **bijection** from set A to set B . The **inverse function of f** is the function that assigns to an element b from B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1} . Hence, $f^{-1}(b) = a$, when $f(a) = b$. If the inverse function of f exists, f is called **invertible**.



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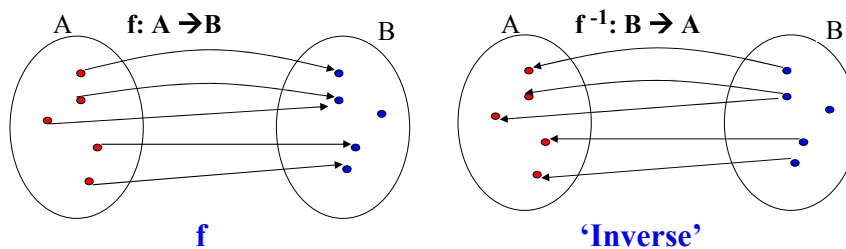
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Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f . **Why?**

Assume f is not one-to-one:

?



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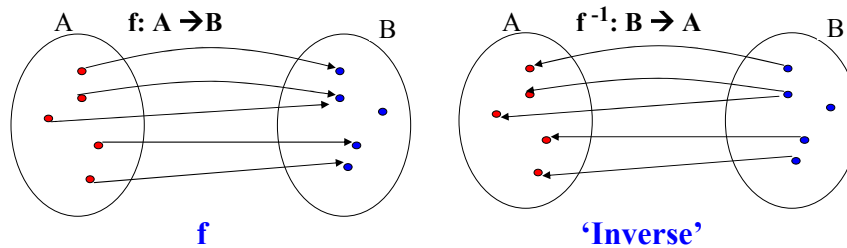
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Inverse functions

Note: if f is not a bijection then it is not possible to define the inverse function of f . **Why?**

Assume f is not one-to-one:

Inverse is not a function. One element of B is mapped to two different elements.



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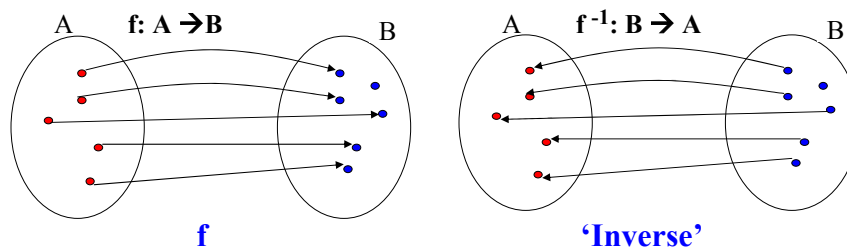
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Note: if f is not a bijection then it is not possible to define the inverse function of f . **Why?**

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Inverse functions

Example 2:

- Let $g : \mathbf{R} \rightarrow \mathbf{R}$, where $g(x) = 2x - 1$.
- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$$\begin{aligned}y &= 2x - 1 \Rightarrow y + 1 = 2x \\ &\Rightarrow (y+1)/2 = x\end{aligned}$$

- Define $g^{-1}(y) = x = (y+1)/2$

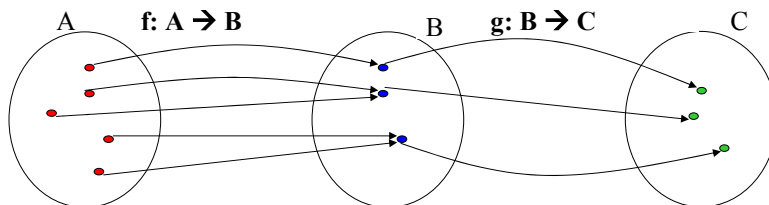
Test the correctness of inverse:

- $g(3) = 2*3 - 1 = 5$
- $g^{-1}(5) = (5+1)/2 = 3$
- $g(10) = 2*10 - 1 = 19$
- $g^{-1}(19) = (19+1)/2 = 10$.

Composition of functions

Definition: Let f be a function from set A to set B and let g be a function from set B to set C . The **composition of the functions g and f** , denoted by $g \circ f$ is defined by

- $(g \circ f)(a) = g(f(a))$.



Composition of functions

Example 1:

- Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$

$$g : A \rightarrow A, \quad f : A \rightarrow B$$

$$1 \rightarrow 3, \quad 1 \rightarrow b$$

$$2 \rightarrow 1, \quad 2 \rightarrow a$$

$$3 \rightarrow 2, \quad 3 \rightarrow d$$

$$f \circ g : A \rightarrow B:$$

- $1 \rightarrow d$
- $2 \rightarrow b$
- $3 \rightarrow a$

Composition of functions

Example 2:

- Let f and g be function from Z into Z , where

- $f(x) = 2x$ and $g(x) = x^2$.

- $f \circ g : Z \rightarrow Z$

- $$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(x^2) \\ &= 2(x^2) \end{aligned}$$

- $g \circ f : Z \rightarrow Z$

- $$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2x) \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

Composition of functions

Example 3:

- $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x .
- Let $f: \mathbf{R} \rightarrow \mathbf{R}$, where $f(x) = 2x - 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x) = f(f^{-1}(x))$
 $= f((x+1)/2)$
 $= 2((x+1)/2) - 1$
 $= (x+1) - 1$
 $= x$
- $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
 $= f^{-1}(2x - 1)$
 $= (2x)/2$
 $= x$

Some functions

Definitions:

- The **floor function** assigns a real number x the largest integer that is less than or equal to x . The floor function is denoted by $\lfloor x \rfloor$.
- The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x . The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

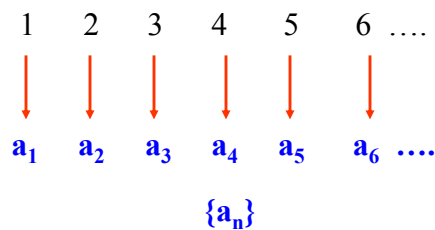
- Factorials: $n! = n(n-1) \dots 1$ such that $1! = 1$

Sequences and summations

Sequences

Definition: A **sequence** is a function from a subset of the set of integers (typically the set $\{0,1,2,\dots\}$ or the set $\{1,2,3,\dots\}$) to a set S . We use the notation a_n to denote the image of the integer n . We call a_n a term of the sequence.

Notation: $\{a_n\}$ is used to represent the sequence (note $\{\}$ is the same notation used for sets, so be careful). $\{a_n\}$ represents the ordered list a_1, a_2, a_3, \dots .



Sequences

Examples:

- (1) $a_n = n^2$, where $n = 1, 2, 3, \dots$
 - What are the elements of the sequence?
1, 4, 9, 16, 25, ...
- (2) $a_n = (-1)^n$, where $n = 0, 1, 2, 3, \dots$
 - Elements of the sequence?
1, -1, 1, -1, 1, ...
- 3) $a_n = 2^n$, where $n = 0, 1, 2, 3, \dots$
 - Elements of the sequence?
1, 2, 4, 8, 16, 32, ...

Arithmetic progression

Definition: An **arithmetic progression** is a sequence of the form
 $a, a+d, a+2d, \dots, a+nd$

where a is the *initial term* and d is *common difference*, such that both belong to \mathbb{R} .

Example:

- $s_n = -1 + 4n$ for $n = 0, 1, 2, 3, \dots$
- members: -1, 3, 7, 11, ...

Geometric progression

Definition A **geometric progression** is a sequence of the form:

$$a, ar, ar^2, \dots, ar^k,$$

where a is the *initial term*, and r is the *common ratio*. Both a and r belong to \mathbb{R} .

Example:

- $a_n = \left(\frac{1}{2}\right)^n$ for $n = 0, 1, 2, 3, \dots$
members: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

Example:

- Assume the sequence: $1, 3, 5, 7, 9, \dots$
- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.
- $1, 1+2=3, 3+2=5, 5+2=7$
- It suggests **an arithmetic progression**: $a+nd$
with $a=1$ and $d=2$
 - $a_n=1+2n$ or $a_n=1+2n$

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

Example 2:

- Assume the sequence: $1, 1/3, 1/9, 1/27, \dots$
- What is the sequence?
- The denominators are powers of 3.
 $1, 1/3 = 1/3, (1/3)/3 = 1/(3 \cdot 3) = 1/9, (1/9)/3 = 1/27$
- This suggests a **geometric progression**: ar^k
with $a=1$ and $r=1/3$
 - $(1/3)^n$