CS 441 Discrete Mathematics for CS
Lecture 3

Propositional logic: Equivalences
Predicate logic

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Propositional logic: review

- **Propositional logic:** a formal language for making logical inferences
- **A proposition is** a statement that is either true or false.
- **A compound proposition** can be created from other propositions using logical connectives
- **The truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- **The truth table for a compound proposition:** table with entries (rows) for all possible combinations of truth values of elementary propositions.

Tautology and Contradiction

Definitions:

- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency**.
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Example: $p \vee \neg p$ is a **tautology.**

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

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Example: $p \wedge \neg p$ is a **contradiction.**

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Equivalence

- Some propositions may be equivalent. Their truth values in the truth table are the same.
- Example: $\mathbf{p \rightarrow q}$ is equivalent to $\mathbf{\neg q \rightarrow \neg p}$ **(contrapositive)**

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T		
T	F		
F	T		
F	F		

Equivalence

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p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- **Equivalent statements** are important for logical reasoning since they can be substituted and can help us to make a logical argument.

Logical equivalence

- **Definition**: The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table). The notation **p <=> q** denotes p and q are logically equivalent.

Examples of equivalences:

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \iff \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \iff \neg p \vee \neg q$

Equivalence

Example of important equivalences

- **DeMorgan's Laws:**
- 1) $\neg(p \vee q) \iff \neg p \wedge \neg q$
- 2) $\neg(p \wedge q) \iff \neg p \vee \neg q$

Use the truth table to prove that the two propositions are logically equivalent

p	q	¬p	¬q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Important logical equivalences

- **Identity**
  - $p \wedge T \iff p$
  - $p \vee F \iff p$

- **Domination**
  - $p \vee T \iff T$
  - $p \wedge F \iff F$

- **Idempotent**
  - $p \vee p \iff p$
  - $p \wedge p \iff p$

Important logical equivalences

- **Double negation**
  - $\neg(\neg p) \iff p$

- **Commutative**
  - $p \vee q \iff q \vee p$
  - $p \wedge q \iff q \wedge p$

- **Associative**
  - $(p \vee q) \vee r \iff p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$

Important logical equivalences

- **Distributive**
  - $p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$
  - $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$

- **De Morgan**
  - $\neg(p \lor q) \Leftrightarrow \neg p \land \neg q$
  - $(p \land q) \Leftrightarrow \neg p \lor \neg q$

- **Other useful equivalences**
  - $p \lor \neg p \Leftrightarrow T$
  - $p \land \neg p \Leftrightarrow F$
  - **$p \rightarrow q \Leftrightarrow (\neg p \lor q)$**

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

Example: Show that $(p \land q) \rightarrow p$ is a tautology.

- Proof: (we must show $(p \land q) \rightarrow p \Leftrightarrow T$)

$(p \land q) \rightarrow p$	$\Leftrightarrow \neg(p \land q) \lor p$	Useful
	$\Leftrightarrow [\neg p \lor \neg q] \lor p$	DeMorgan
	$\Leftrightarrow [\neg q \lor \neg p] \lor p$	Commutative
	$\Leftrightarrow \neg q \lor [\neg p \lor p]$	Associative
	$\Leftrightarrow \neg q \lor [T]$	Useful
	$\Leftrightarrow T$	Domination

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

Example: Show $(p \wedge q) \rightarrow p$ is a tautology.

- Alternative proof:

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

Using logical equivalences

- **Equivalences can be used in proofs.** A proposition or its part can be transformed using equivalences and some conclusion can be reached.

- **Proofs that rely on logical equivalences can replace truth table approach**
  - **Why?**
  - The truth table has 2^n rows, where n is the number of elementary propositions
  - If n is large building the truth table may become infeasible

Using logical equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences and some conclusion can be reached.

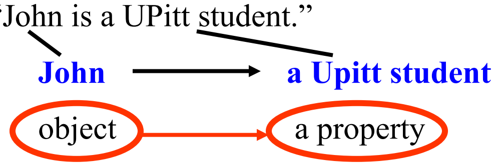
- **Example 2:** Show $(p \rightarrow q) <=> (\neg q \rightarrow \neg p)$

  Proof:
- $(p \rightarrow q) <=> (\neg q \rightarrow \neg p)$
- $<=> \neg(\neg q) \vee (\neg p)$ Useful
- $<=> q \vee (\neg p)$ Double negation
- $<=> \neg p \vee q$ Commutative
- $<=> p \rightarrow q$ Useful

  End of proof

Limitations of the propositional logic

- **Propositional logic:** the world is described in terms of propositions
- **A proposition is** a statement that is either true or false.
- **Limitations:**
  - **objects** in elementary statements, their properties and relations are not explicitly represented in the propositional logic
- **Example:**
  - "John is a UPitt student."



  - Objects and properties are hidden in the statement, it is not possible to reason about them

Limitations of the propositional logic

- **Statements for groups of objects**
  – In propositional logic these must be exhaustively enumerated
- **Example:**
  – If John is a CS UPitt graduate then John has passed cs441

  Translation:

  – John is a CS UPitt graduate → John has passed cs441

  Similar statements can be written for other Upitt graduates:

  – Ann is a CS Upitt graduate → Ann has passed cs441

  – Ken is a CS Upitt graduate → Ken has passed cs441

  – …
- **What is a more natural solution to express the above knowledge?**

Limitations of the propositional logic

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  – In propositional logic these must be exhaustively enumerated
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  Translation:

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  Similar statements can be written for other Upitt graduates:

  – Ann is a CS Upitt graduate → Ann has passed cs441

  – Ken is a CS Upitt graduate → Ken has passed cs441

  – …
- **Solution:** make statements with **variables**
  – If x is a CS Upitt graduate then x has passed cs441
  – x is a CS UPitt graduate → x has passed cs441

Predicate logic

Remedies the limitations of the propositional logic

- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

Basic building blocks of the predicate logic:

- **Constant** –models a specific object
  Examples: "John", "France", "7"
- **Variable** – represents object of specific type **(defined by the universe of discourse)**
  Examples: x, y
  (universe of discourse can be people, students, numbers)
- **Predicate -** over one, two or many variables or constants.
  – Represents properties or relations among objects
  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

A predicate P(x) assigns a value **true or false** to each x depending on whether the property holds or not for x.

- The assignment is best viewed as a big table with the variable x substituted for objects from the universe of discourse

Example:

- Assume **Student(x)** where the universe of discourse are people
- Student(John) …. T (if John is a student)
- Student(Ann) …. T (if Ann is a student)
- Student(Jane) ….. F (if Jane is not a student)
- …

Predicates

Assume a predicate P(x) that represents the statement:

- **x is a prime number**

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

All statements P(2), P(3), P(4), P(5), P(6), P(7) are propositions

Predicates

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What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T
- P(6) F
- P(7) T

Is P(x) a proposition? No. Many possible substitutions are possible.

Predicates

Important:

- predicate P(x) is **not a proposition** since there are more objects it can be applied to

This is the same as in propositional logic …

… But the difference is:

- propositional logic does not let us go inside the statements and manipulate x
- predicate logic allows us to explicitly manipulate and substitute for the objects

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

Example:

- Older(John, Peter) denotes 'John is older than Peter'
  - this is a proposition because it is either true or false
- Older(x,y) - 'x is older than y'
  - not a proposition, but after the substitution it becomes one

Predicates

- Predicates can have **more arguments** which represent the **relations between objects**

Example:

- Let Q(x,y) denote 'x+5 >y'
  - Is Q(x,y) a proposition? **No!**
  - Is Q(3,7) a proposition? **Yes.** It is true.
  - What is the truth value of
  - Q(3,7) T
  - Q(1,6) F
  - Q(2,2) T
  - Is Q(3,y) a proposition? **No!** We cannot say if it is true or false.

Quantified statements

Predicate logic allows us to make statements about groups of objects

- To do this we use special quantified expressions

Two types of quantified statements:

- **universal**

  Example: ' all CS Upitt graduates have to pass cs441"
  - the statement is true for all graduates
- **existential**

  Example: 'Some CS Upitt students graduate with honor.'
  - the statement is true for some people

Universal quantifier

Defn: The universal quantification of P(x) is the proposition "P(x) is true for all values of x in the universe of discourse." The notation **∀x P(x)** denotes the universal quantification of P(x), and is expressed as **for every x, P(x).**

Example:

- Let P(x) denote x > x - 1.
- What is the truth value of ∀x P(x)?
- Assume the universe of discourse of x is all real numbers.
- **Answer:** Since every number x is greater than itself minus 1. Therefore, **∀x P(x) is true.**

Universal quantifier

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Example 2:

- Let T(x) denote x > 5.
- What is the truth value of ∀x T(x)?
- Assume the universe of discourse of x is all real numbers.
- **Answer:**
  - Since 3 > 5 is false. So, T(x) is not true for all values of x. Therefore, it is **false that ∀x T(x).**

Universal quantifier

Quantification converts a propositional function into **a proposition** by binding a variable to a set of values from the universe of discourse.

Example:

- Let P(x) denote x > x - 1.
- Is P(x) a proposition? **No.** Many possible substitutions.
- Is $\forall$x P(x) a proposition? **Yes.** True if for all x from the universe of discourse P(x) is true. Which holds?
- Is $\forall$x Q(x,y) a proposition? **No.** The variable y is free and can be substituted by many objects.