CS 441 Discrete Mathematics for CS
Lecture 24

Relations IV

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive.**

Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive**.

Example: Let A = {0,1,2,3,4,5,6} and

- R= {(a,b)| a,b $\in$ A, a $\equiv$ b mod 3} (a is congruent to b modulo 3)

Congruencies:

- 0 mod 3 = 0 1 mod 3 = 1 2 mod 3 = 2 3 mod 3 = 0
- 4 mod 3 = 1 5 mod 3 = 2 6 mod 3 = 0

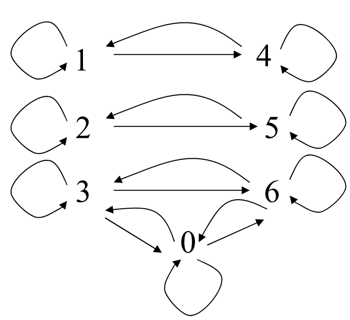
Relation R has the following pairs:

- (0,0) (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
- (2,2), (2,5), (5,2), (5,5)

Equivalence relation

- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**
  (0,0) (0,3), (3,0), (0,6), (6,0)
  (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
  (2,2), (2,5), (5,2), (5,5)

- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive. **Yes.**



Then

- **R is an equivalence relation**.

Equivalence class

Definition: Let R be an equivalence relation on a set A. The set $\{ x \in A \mid a R x\}$ is called **the equivalence class of a,** denoted by $[a]_R$ or simply [a]. If $b \in [a]$ then b is called **a representative of this equivalence class**.

Example:

- Assume R={(a,b) | a ≡ b mod 3} for A={0,1,2,3,4,5,6}

R= {(0,0), (0,3), (3,0), (0,6), (6,0),(3,3), (3,6) (6,3), (6,6), (1,1),(1,4), (4,1), (4,4), (2,2), (2,5), (5,2), (5,5)}

- **Pick an element** a =0.
- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R= \{1,4\}$
- Element 2: $[2]_R= \{2,5\}$
- Element 3: $[3]_R= \{0,3,6\} = [0]_R = [6]_R$
- Element 4: $[4]_R= \{1,4\} = [1]_R$ Element 5: $[5]_R= \{2,5\} = [2]_R$

Equivalence class

Example:

- Assume R={(a,b) | a ≡ b mod 3} for A={0,1,2,3,4,5,6}
- R= {(0,0), (0,3), (3,0), (0,6), (6,0),(3,3), (3,6) (6,3), (6,6), (1,1),(1,4), (4,1), (4,4), (2,2), (2,5), (5,2), (5,5)}

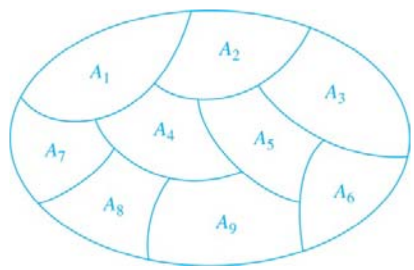
Three different equivalence classes all together:

- $[0]_R = [3]_R =[6]_R = \{0,3,6\}$
- $[1]_R= [4]_R= \{1,4\}$
- $[2]_R= [5]_R= \{2,5\}$

Partition of a set S

Definition: Let S be a set. A collection of nonempty subsets of S $A_1, A_2, ..., A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \emptyset, \ i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$



Partition of a set S

Definition: Let S be a set. A collection of nonempty subsets of S $A_1, A_2, ..., A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \emptyset, \ i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$

Example: Let S={1,2,3,4,5,6} and

- A_1 ={0,3,6} A_2 ={1,4} A_3 = {2,5}
- Is A_1, $A_{2,,}$ A_3 a partition of S ? **Yes.**

- Give a partition of S ?
- {0,2,4,6} {1,3,5}
- {0} {1,2} {3,4,5} {6}

Equivalence classes and partitions

Theorem: Let R be an equivalence relation on a set A. Then the union of all the equivalence classes of R is A:

$$A = \bigcup_{a \in A}^{k} [a]_R$$

Proof: an element a of A is in its own equivalence class $[a]_R$ so union cover A.

Theorem: The equivalence classes form a partition of A.

Proof: The equivalence classes split A into disjoint subsets.

Theorem : Let $\{A_1, A_2, .. A_i ,..\}$ be a partitioning of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Partial orderings

Definition: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Example: Assume R denotes the "greater than or equal" relation ($\geq$) on the set S={1,2,3,4,5}.

- Is the relation reflexive? Yes
- Is it antisymmetric? Yes
- Is it transitive? Yes
- **Conclusion:** R is a partial ordering.

Partial orderings

Example: Assume R is the divisibility relation (|) on the set of integers S={1,2,3,4,5,6}

- Is the relation reflexive? Yes
- Is it antisymmetric? Yes
- Is it transitive? Yes
- **Conclusion:** R is a partial ordering.

Comparability

Definition 1: The elements *a* and *b* of a poset $(S,\preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When *a* and *b* are elements of *S* so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$ holds, then *a* and *b* are called ***in**comparable*.

Definition 2: If $(S,\preccurlyeq)$ is a poset and every two elements of *S* are comparable, *S* is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain.*

Definition 3: $(S,\preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of *S* has a least element.

Lexicographical ordering

Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is, $(a_1, a_2) \prec (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \prec_2 b_2$.

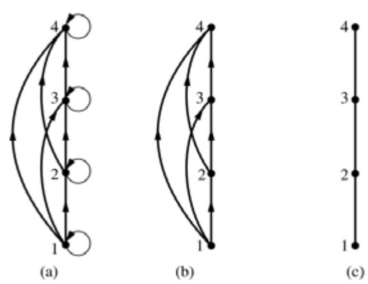
The definition can be extended to a lexicographic ordering on strings

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* ≺ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* ≺ *discreetness*, because the first eight letters agree, but the second string is longer.

Hasse diagram

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



(a) A partial ordering. The loops are due to the reflexive property

(b) The edges that must be present due to the transitive property are deleted

(c) The Hasse diagram for the partial ordering (a).

Procedure for constructing Hasse diagram

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:
  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

CS 441 Discrete Mathematics for CS
Lecture 24b

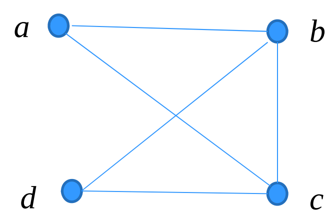
Graphs
(chapter 10)

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Definition of a graph

- **Definition:** A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges.* Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

- **Example:**

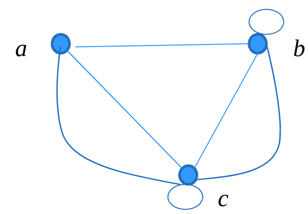


Terminology

- In a ***simple graph*** each edge connects two different vertices and no two edges connect the same pair of vertices.
- ***Multigraphs*** may have multiple edges connecting the same two vertices. When m different edges connect the vertices u and v, we say that $\{u,v\}$ is an edge of *multiplicity m*.
- An edge that connects a vertex to itself is called a ***loop***.
- A ***pseudograph*** may include loops, as well as multiple edges connecting the same pair of vertices.

Terminology

- In a *simple graph* each edge connects two different vertices and no two edges connect the same pair of vertices.
- *Multigraphs* may have multiple edges connecting the same two vertices. When *m* different edges connect the vertices *u* and *v*, we say that $\{u,v\}$ is an edge of *multiplicity m*.
- An edge that connects a vertex to itself is called a *loop*.
- A *pseudograph* may include loops, as well as multiple edges connecting the same pair of vertices.



Directed graph

Definition: An *directed graph* (or *digraph*) $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *directed edges* (or *arcs*). Each edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u,v) is said to *start at u* and *end at v*.

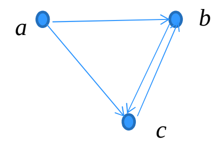
Remark:

– Graphs where the end points of an edge are not ordered are said to be ***undirected graphs.***

Directed graph

- A *simple directed graph* has no loops and no multiple edges.

Example:



- A *directed multigraph* may have multiple directed edges. When there are *m* directed edges from the vertex *u* to the vertex *v*, we say that (*u*,*v*) is an edge of *multiplicity m*.

Example:

- multiplicity of (*a*,*b*) is ?
- and the multiplicity of (*b*,*c*) is ?