CS 441 Discrete Mathematics for CS
Lecture 23

Relations III.

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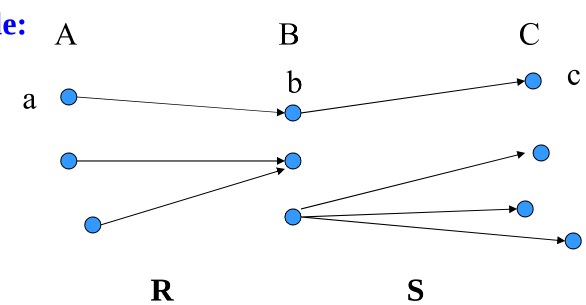
Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where a ∈ A and c ∈ C, and for which there is a b ∈ B such that (a,b) ∈ R and (b,c) ∈ S. We denote the composite of R and S by S o R.

Example:



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(a,c) ∈ S o R

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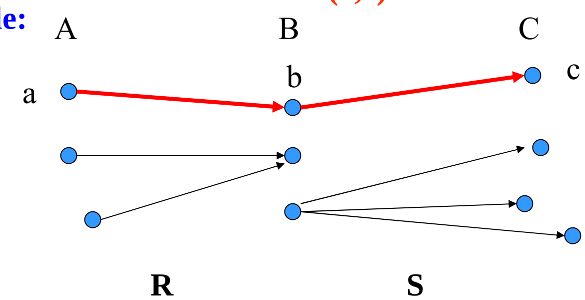

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Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = ?

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- S o R = {(1,b),(3,a),(3,b)}

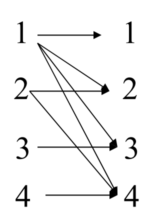
Representing binary relations with graphs

- We can graphically represent a binary relation R from A to B as follows:
  - if **a R b** then draw an arrow from a to b.

a → b

Example:

- Relation R_{div} (from previous lectures) on A={1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}



Representing relations on a set with digraphs

Definition: A **directed graph or digraph** consists of a set of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b) and vertex b is the terminal vertex of this edge. An edge of the form (a,a) is called **a loop**.

Example

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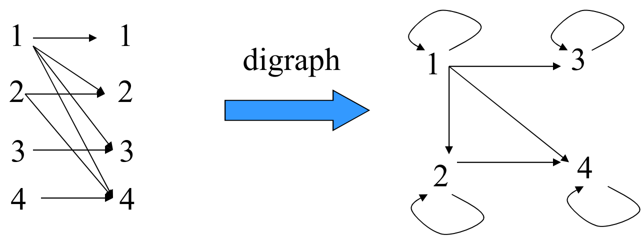

Powers of R

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \circ R}$.

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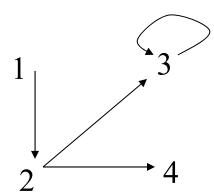

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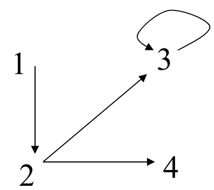

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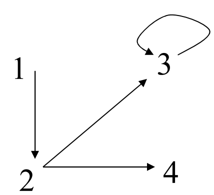

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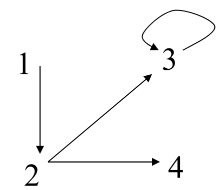

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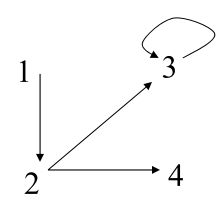

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- R^3 = {(1,3), (2,3), (3,3)} path of length 3

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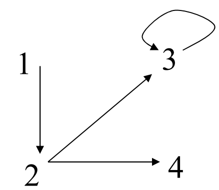

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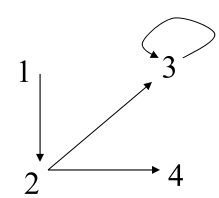

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- R^4 = {(1,3), (2,3), (3,3)}
- R^k = {(1,3), (2,3), (3,3)} k>3

Transitive relation and R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n = 1,2,3,... .

Proof: bi-conditional (if and only if)

Proved last lecture

Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: ?

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- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation** **S** $\supseteq$ **R** that is reflexive?
- How to make R reflexive with minimum number of additions?
- Answer: ?

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- Is this relation reflexive?
- Answer: **No.** Why?
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- The question is what is **the minimal relation S** $\supseteq$ R that is reflexive?
- How to make R reflexive with minimum number of additions?
- **Answer:** Add (2,2) and (3,3)
  - Then S= {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
  - R $\subseteq$ S
  - The minimal set S $\supseteq$ R is called **the reflexive closure of R**

Reflexive closure

The set S is called **the reflexive closure of R** if it:

- contains R
- has reflexive property
- is contained in every reflexive relation Q that contains R (R $\subseteq$ Q) , that is S $\subseteq$ Q

Closures on relations

- **Relations can have different properties:**
  - **reflexive,**
  - **symmetric**
  - **transitive**

- Because of that we define:
  - **symmetric,**
  - **reflexive and**
  - **transitive**

  closures.

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

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Example (symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S=?

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Example (transitive closure):

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- **Is R transitive?**

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Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} $\cup$ {(1,3)}**
  = {(1,2), (2,2), (2,3),(1,3)}
- S is the transitive closure of R

Transitive closure

We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

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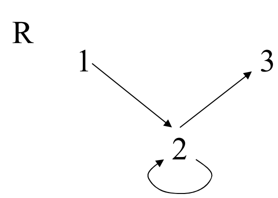

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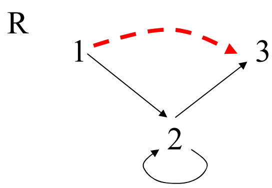

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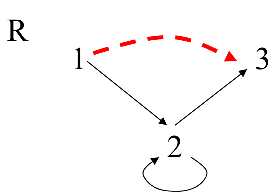
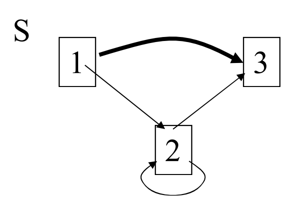
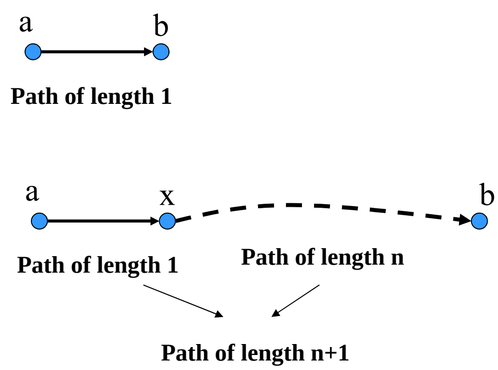




Path length

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):



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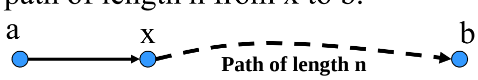
- There is a path of length n+1 from a to b if and only if there exists an $x \in A$, such that $(a,x) \in R$ (a path of length 1) and $(x,b) \in R^n$ is a path of length n from x to b.



- $(x,b) \in R^n$ holds due to P(n). Therefore, there is a path of length n + 1 from a to b. This also implies that $(a,b) \in R^{n+1}$.

Connectivity relation

Definition: Let R be a relation on a set A. The **connectivity relation** R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^* = \bigcup_{k=1}^{\infty} R^k$$

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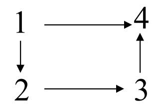

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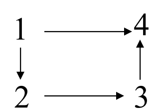

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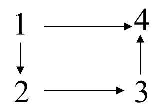

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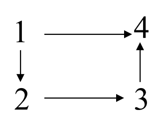

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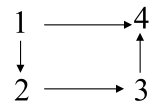

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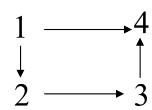

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- ...
- R* = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

Connectivity

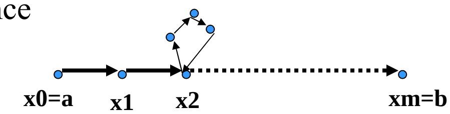
Lemma 1: Let A be a set with n elements, and R a relation on A. If there is a path from a to b, then there exists a path of length < n in between (a,b). Consequently:

$$R^* = \bigcup_{k=1}^{n} R^k$$

Proof (intuition):

- There are at most n different elements we can visit on a path if the path does not have loops

- Loops may increase the length but the same node is visited more than once



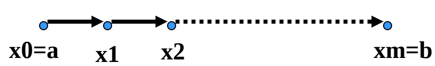
Connectivity

Lemma 1: Let A be a set with n elements, and R a relation on A. If there is a path from a to b, then there exists a path of length < n in between (a,b). Consequently:

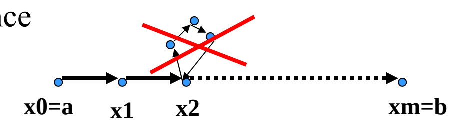
$$R^* = \bigcup_{k=1}^{n} R^k$$

Proof (intuition):

- There are at most n different elements we can visit on a path if the path does not have loops



- Loops may increase the length but the same node is visited more than once



Transitivity closure and connectivity relation

Theorem: The transitive closure of a relation R **equals** the connectivity relation R*.

Based on the **Lemma 1.**

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Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive.**

Example: Let A = {0,1,2,3,4,5,6} and

- R= {(a,b)| a,b $\in$ A, a $\equiv$ b mod 3} (a is congruent to b modulo 3)

Congruencies:

- 0 mod 3 = ?

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- 0 mod 3 = 0 1 mod 3 = 1 2 mod 3 = 2 3 mod 3 = 0
- 4 mod 3 = ?

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Relation R has the following pairs:

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Relation R has the following pairs:

- (0,0) (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)
- (2,2), (2,5), (5,2), (5,5)

Equivalence relation

- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**
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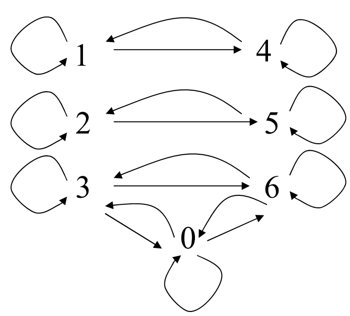
- Is R reflexive?



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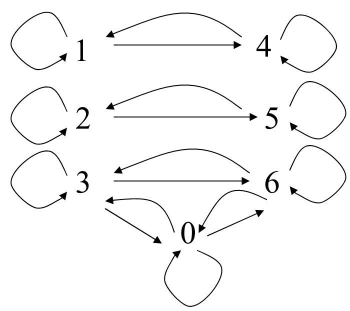
- Is R reflexive? **Yes.**
- Is R symmetric?



Equivalence relation

- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**

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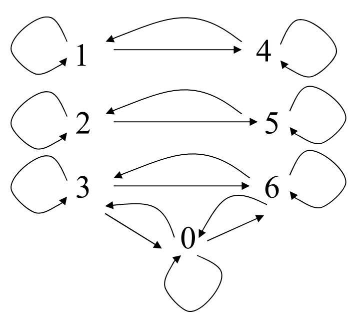
- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive?



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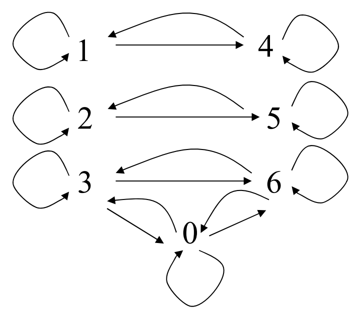
- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive. **Yes.**



Then

- **R is an equivalence relation**.

Equivalence class

Theorem: Let R be an **equivalence relation** on a set A. The following statements are equivalent:

- i) a R b
- ii) [a] = [b]
- iii) $[a] \cap [b] \neq \emptyset$.

Proof: (iii) $\rightarrow$ (i)

- Suppose $[a] \cap [b] \neq \emptyset$, want to show a R b.
- $[a] \cap [b] \neq \emptyset \rightarrow x \in [a] \cap [b] \rightarrow x \in [a]$ and $x \in [b] \rightarrow (a,x)$ and $(b,x) \in R$.
- Since R is symmetric $(x,b) \in R$. By the transitivity of R $(a,x) \in R$ and $(x,b) \in R$ implies $(a,b) \in R \rightarrow$ a R b.