CS 441 Discrete Mathematics for CS
Lecture 22

Relations II

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Cartesian product (review)

- Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.
- **The Cartesian product** A x B is defined by a set of pairs $\{(a_1\ b_1), (a_1, b_2), \dots (a_1, b_m), \dots, (a_k,b_m)\}$.

Example:

Let A={a,b,c} and B={1 2 3}. What is AxB?

AxB = {(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)}

Binary relation

Definition: Let A and B be sets. A **binary relation from A to B** is **a subset of a Cartesian product A x B**.

Example: Let A={a,b,c} and B={1,2,3}.

- R={(a,1),(b,2),(c,2)} is an example of a relation from A to B.

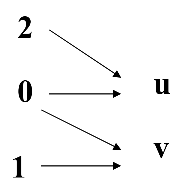
Representing binary relations

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

a → b

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- Note: R ⊆ A x B.
- **Graph:**



Representing binary relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- **Table:**

R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

Properties of relations

Properties of relations on A:

- Reflexive ✓
- Irreflexive ✓
- Symmetric ✓
- **Anti-symmetric**
- **Transitive**

Reflexive relation

Reflexive relation

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- **A relation R is reflexive** if and only if MR has 1 in every position on its main diagonal.

Irreflexive relation

Irreflexive relation

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a ≠ b.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is irreflexive** if and only if MR has 0 in every position on its main diagonal.

Symmetric relation

Symmetric relation:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is symmetric** if and only if $m_{ij} = m_{ji}$ for all i,j.

Anti-symmetric relation

Definition (anti-symmetric relation): A relation on a set A is called **anti-symmetric** if

- $[(a,b) \in R$ and $(b,a) \in R] \rightarrow a = b$ where $a, b \in A$.

Example 3:

- Relation R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} anti-symmetric?**
- **Answer: Yes.** It is anti-symmetric

Anti-symmetric relation

Antisymmetric relation

- relation R_{fun} = {(1,2),(2,2),(3,3)}

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- A relation is **antisymmetric** if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**
- **Answer:**

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R$ and $(b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- $R_{div} = \{(a\ b),$ if $a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer: Yes.**

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R$ and $(b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on $A=\{1,2,3,4\}$, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- $R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$
- **Is $R_{\neq}$ transitive ?**
- **Answer:**

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Is $R_{\neq}$ transitive?**
- **Answer: No.** It is not transitive since $(1,2) \in R$ and $(2,1) \in R$ but (1,1) is not an element of R.

Transitive relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example** 3:
- Relation R_{fun} on A = {1,2,3,4} defined as:
  - R_{fun} = {(1,2),(2,2),(3,3)}.
- **Is R_{fun} transitive?**
- **Answer:**

Transitive relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example** 3:
- Relation R_{fun} on A = {1,2,3,4} defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} transitive?**
- **Answer: Yes.** It is transitive.

Combining relations

Definition: Let A and B be sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.

Combining Relations

- **Relations are sets → combinations via set operations**
- Set operations of: **union, intersection, difference and symmetric difference.**

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- $R1 \cup R2 = ?$

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- $R1 \cup R2$ = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- $R1 \cap R2 = ?$

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 ∪ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 ∩ R2 = {(3,u)}
- R1 - R2 = ?

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 ∪ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 ∩ R2 = {(3,u)}
- R1 - R2 = {(1,u),(2,u),(2,v)}
- R2 - R1 = ?

Combining relations

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 $\cup$ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 $\cap$ R2 = {(3,u)}
- R1 - R2 = {(1,u),(2,u),(2,v)}
- R2 - R1 = {(1,v),(3,v)}

Combination of relations

Representation of operations on relations:

- **Question:** Can the relation be formed by taking the union or intersection or composition of two relations R1 and R2 be represented in terms of matrix operations?
- **Answer: Yes**

Combination of relations: implementation

Definition. The **join,** denoted by $\vee$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \vee b_{ij}$ for all i,j

  = pairwise or (disjunction)

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- **MR1** = $\begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix}$ **MR2** = $\begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix}$ **M(R1** $\vee$ **R2)** = $\begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}$

Combination of relations: implementation

Definition. The **meet,** denoted by $\wedge$, of two m-by-n matrices (a_{ij}) and (b_{ij}) of 0s and 1s is an m-by-n matrix (m_{ij}) where

- $m_{ij} = a_{ij} \wedge b_{ij}$ for all i,j

  = **pairwise and (conjunction)**

- **Example**:
- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

- **MR1** = $\begin{matrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{matrix}$ **MR2** = $\begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix}$ **MR1** $\wedge$ **MR2** = $\begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{matrix}$

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where a ∈ A and c ∈ C, and for which there is a b ∈ B such that (a,b) ∈ R and (b,c) ∈ S. We denote the composite of R and S by S o R.

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = ?

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where a ∈ A and c ∈ C, and for which there is a b ∈ B such that (a,b) ∈ R and (b,c) ∈ S. We denote the composite of R and S by S o R.

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = {(1,b),(3,a),(3,b)}

Implementation of composite

Definition. The **Boolean product**, **denoted by** $\odot$**,** of an m-by-n matrix (a_{ij}) and n-by-p matrix (b_{jk}) of 0s and 1s is an m-by-p matrix (m_{ik}) where

- m_{ik} = 1, if $a_{ij} = 1$ and $b_{jk} = 1$ for some k=1,2,...,n
  0, otherwise

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}
- S o R = {(1,b),(3,a),(3,b)}

Implementation of composite

Examples:

- Let A = {1,2}, B= {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$M_R \odot M_S$ = ?

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
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$$M_R \odot M_S = \begin{matrix} x & x \\ x & x \end{matrix}$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
- S = {(1,a),(3,b),(3,a)} is a relation from B to C.
- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad M_S = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{matrix}$$

$$M_R \odot M_S = \begin{matrix} 1 & x \\ x & x \end{matrix}$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
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Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
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$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M_R \odot M_S = \begin{bmatrix} 1 & 1 \\ 1 & x \end{bmatrix}$$

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
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- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M_R \odot M_S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$M_{S \circ R} =$?

Implementation of composite

Examples:

- Let A = {1,2}, {1,2,3} C = {a,b}
- R = {(1,2),(1,3),(2,1)} is a relation from A to B
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- S ○ R = {(1,b),(1,a),(2,a)}

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M_R \odot M_S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M_{S \circ R} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \circ R}$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = ?

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

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Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = ?

Composite of relations

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Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = ?

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

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Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = ?

Composite of relations

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Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- R^k = ? , k > 3.

Composite of relations

Definition: Let R be a relation on a set A. The **powers R^n**, n = 1,2,3,... is defined inductively by

- $\mathbf{R^1 = R}$ and $\mathbf{R^{n+1} = R^n \circ R}$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = R = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- $R^k = R^3$, k > 3.

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- [(a,b) ∈ R and (b,c) ∈ R] → (a,c) ∈ R for all a, b, c ∈ A.

- **Example 1:**
- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**
- **Answer: ?**

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- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**
- **Answer: Yes.**

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n = 1,2,3,... .

Proof: biconditional (if and only if)

(←) Suppose $R^n \subseteq R$, for n =1,2,3,... .

- Let $(a,b) \in R$ and $(b,c) \in R$
- by the definition of $R \circ R$, $(a,c) \in R \circ R = R^2 \subseteq R$
- Therefore R is transitive.

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n = 1,2,3,... .

Proof: biconditional (if and only if)

(→) Suppose R is transitive. Show $R^n \subseteq R$, for n =1,2,3,... .

- Let $P(n) : R^n \subseteq R$. Mathematical induction.
- **Basis Step:**

Connection to R^n

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n = 1,2,3,... .

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(→) Suppose R is transitive. Show $R^n \subseteq R$, for n =1,2,3,... .

- Let P(n) : $R^n \subseteq R$. Mathematical induction.
- **Basis Step:** P(1) says $R^1 = R$ so, $R^1 \subseteq R$ is true.

Connection to R^n

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- Let P(n) : $R^n \subseteq R$. Mathematical induction.
- **Basis Step:** P(1) says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.

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(→) Suppose R is transitive. Show $R^n \subseteq R$, for n =1,2,3,... .

- Let P(n) : $R^n \subseteq R$. Mathematical induction.
- **Basis Step:** P(1) says $R^1 = R$ so, $R^1 \subseteq R$ is true.
- **Inductive Step:** show $P(n) \rightarrow P(n+1)$
- Want to show if $R^n \subseteq R$ then $R^{n+1} \subseteq R$.
- Let $(a,b) \in R^{n+1}$ then by the definition of $R^{n+1} = R^n \circ R$ there is an element $x \in A$ so that $(a,x) \in R$ and $(x,b) \in R^n \subseteq R$ (inductive hypothesis). In addition to $(a,x) \in R$ and $(x,b) \in R$, R is transitive; so $(a,b) \in R$.
- Therefore, $R^{n+1} \subseteq R$.

Number of reflexive relations

Theorem: The number of reflexive relations on a set A, where $|A| = n$ is: $2^{n(n-1)}$.

Proof:

- A reflexive relation R on A **must contain** all pairs (a,a) where $a \in A$.
- All other pairs in R are of the form (a,b), $a \neq b$, such that $a, b \in A$.
- How many of these pairs are there? Answer: n(n-1).
- How many subsets on n(n-1) elements are there?
- **Answer:** $2^{n(n-1)}$.