CS 441 Discrete Mathematics for CS
Lecture 21

Probabilities:
Expected value

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Probability basics

Sample space S: space of all possible outcomes

Event E: a subset of outcomes

Probability: a number in [0,1] we can associate with an an outcome or an event

$$p(E) = \sum_{s \in E} p(s)$$

Probability distribution

A function p: S → [0,1] that assigns a probability to every possible outcome in S

$$\sum_{s \in S} p(s) = 1$$

Conditional probability

Definition: Let E and F be two events such that $P(F) > 0$. The **conditional probability** of E given F

- $P(E| F) = P(E| F) = P(E \cap F) / P(F)$

Corrolary: Let E and F be two events such that $P(F) > 0$. Then:
- $P(E \cap F) = P(E| F) * P(F)$

   This result is also referred to as a product rule.

Bayes theorem

Definition: Let E and F be two events such that $P(F) > 0$. Then:
- $P(E| F) = P(F| E)P(E) / P(F)$
   $= P(F|E)P(E) / (P(F|E)P(E) + P(F|\sim E)P(\sim E))$

- **Expresses P(E | F) in terms of P(F |E)**
- conditioning event are switched

Independence

Definition: The two events E and F are said to be **independent** if:

• $P(E \cap F) = P(E)P(F)$

Random variables

• **A random variable** is a function from the **sample space of an experiment** to the set of real numbers f: S → R. A random variable assigns a number to each possible outcome.

• **The distribution of a random variable X on the sample space S** is a set of pairs (r p(X=r)) for all r in S where r is the number and p(X=r) is the probability that X takes a value r.

Random variables

Example:

Let S be the outcomes of a two-dice roll

Let random variable X denotes the sum of outcomes

(1,1) → 2

(1,2) and (2,1) → 3

(1,3), (3,1) and (2,2) → 4

…

Distribution of X:

- 2 → 1/36,
- 3 → 2/36,
- 4 → 3/36 …
- 12 → 1/36

Expected value

Definition: The expected value of the random variable X(s) on the sample space is equal to:

$$E(X) = \sum_{s \in S} p(s) X(s)$$

Example: roll of a dice

- Outcomes: 1 2 3 4 5 6
- Expected value:

E(X) = 1*1/6 + 2*1/6+3*1/6 + 4*1/6 + 5*1/6 +6* 1/6 = 7/2

Expected value

Investment problem:

- You have 100 dollars and can invest into a stock. The returns are volatile and you may get either $120 with probability of 0.4, or $90 with probability 0.6.
- **What is the expected value of your investment?**
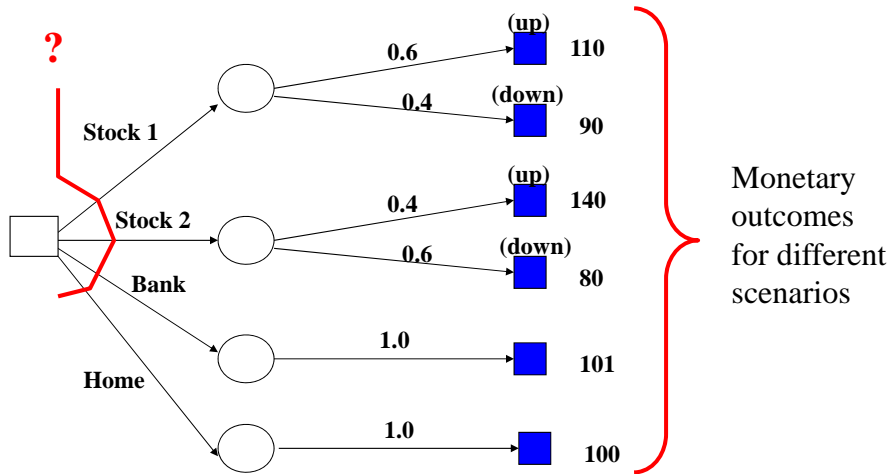-

Expected value

Investment problem:

- You have 100 dollars and can invest into a stock. The returns are volatile and you may get either $120 with probability of 0.4, or $90 with probability 0.6.
- **What is the expected value of your investment?**
- $E(X) = 0.4*120+0.6*90=48+54=102$
- **Is it OK to invest?**

Decision making example

We need to make a choice whether to invest in Stock 1 or 2, put money into bank or keep them at home.

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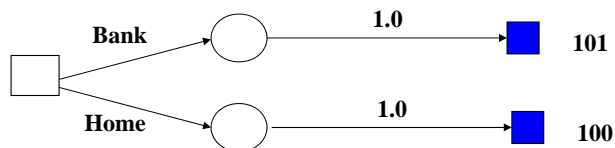Stock 1

0.6 (up) ▪ 110

0.4 (down) ▪ 90

Stock 2

0.4 (up) ▪ 140

0.6 (down) ▪ 80

Bank

1.0 ▪ 101

Home

1.0 ▪ 100

Monetary outcomes for different scenarios

Decision making example.

Assume the simplified problem with the Bank and Home choices only.

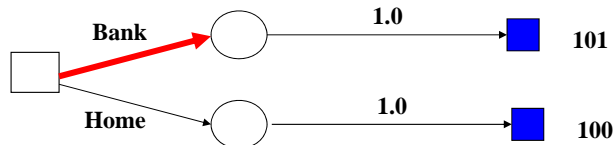**The result is guaranteed – the outcome is deterministic**

Bank

1.0 ▪ 101

Home

1.0 ▪ 100

What is the rational choice assuming our goal is to make money?

Decision making. Deterministic outcome.

Assume the simplified problem with the Bank and Home choices only.

These choices are deterministic.

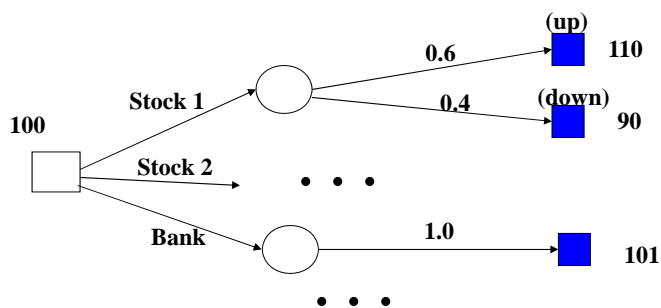Our goal is to make money. What is the rational choice?

Answer: Put money into the bank. The choice is always strictly better in terms of the outcome

Decision making

- **How to quantify the goodness of the stochastic outcome?**
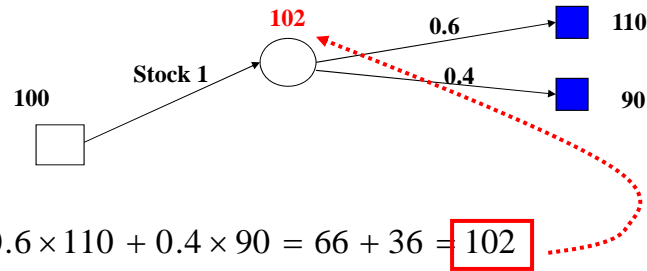 **We want to compare it to deterministic and other stochastic outcomes.**

Idea: <u>Use the expected value of the outcome</u>

Expected value
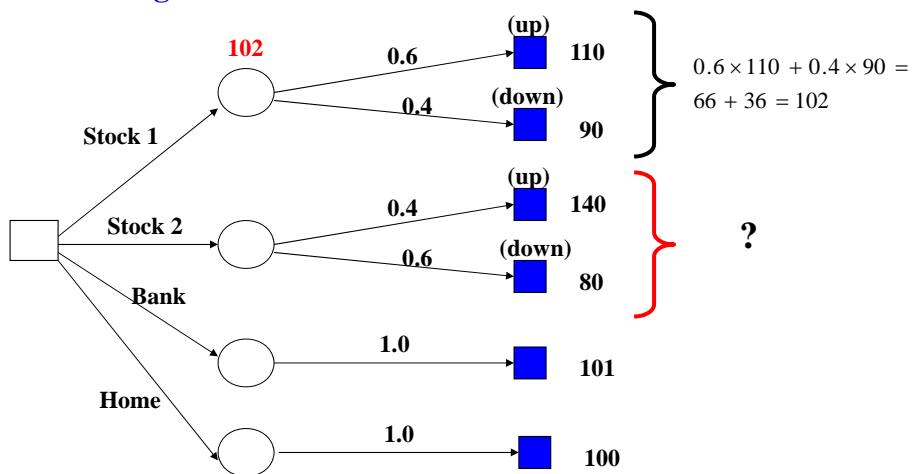
- **Expected value** summarizes all stochastic outcomes into a single quantity
  - Expected value for the outcome of the Stock 1 option is:

102

100

Stock 1

0.6 → 110

0.4 → 90

$$0.6 \times 110 + 0.4 \times 90 = 66 + 36 = \boxed{102}$$

Expected values

Investing \$100 for 6 months

102

Stock 1

0.6 — **(up)** 110

0.4 — **(down)** 90

$$0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102$$

Stock 2

0.4 — **(up)** 140

0.6 — **(down)** 80

?

Bank

1.0 → 101

Home

1.0 → 100

Expected values

Investing \$100 for 6 months

102   0.6   **(up)**   110

Stock 1   0.4   **(down)**   90

$$0.6 \times 110 + 0.4 \times 90 = 66 + 36 = 102$$

104   0.4   **(up)**   140

Stock 2   0.6   **(down)**   80

$$0.4 \times 140 + 0.6 \times 80 = 56 + 48 = 104$$

Bank   1.0   101

Home   1.0   100
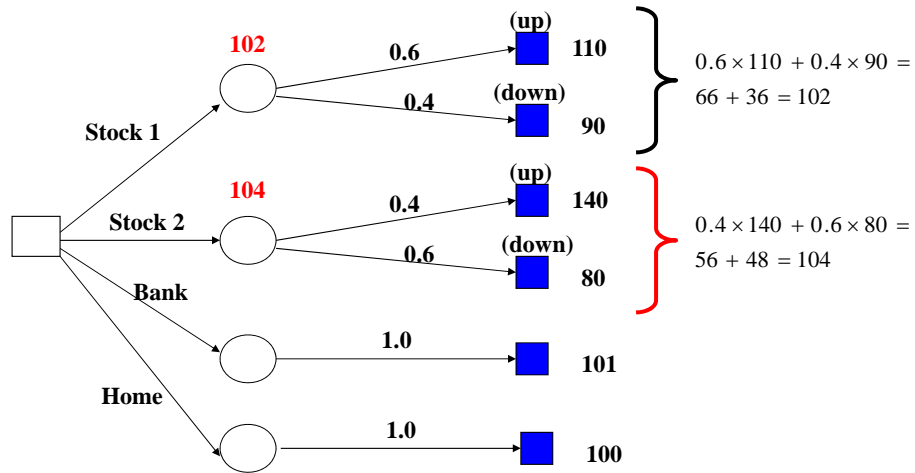
Selection based on expected values

The optimal action is the option that maximizes the expected outcome:

102   0.6   **(up)**   110

Stock 1   0.4   **(down)**   90

104   0.4   **(up)**   140

Stock 2   0.6   **(down)**   80

Bank   **101**   1.0   101

Home   **100**   1.0   100