CS 441 Discrete Mathematics for CS
Lecture 18

Counting

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Permutations

A permutation of a set of distinct objects is an **ordered arrangement** of the objects. Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

The number of permutations of n elements is:

P(n,n) = n!

k-permutations

- ***k*-permutation** is an ordered arrangement of *k* elements of a set.

Example:

- Assume we have a set S with n elements. S={a,b,c}.
- **2 permutations of S:**
- **ab ba ac ca bc cb**

- The number of *k*-permutations of a set with *n* distinct elements is:

  P(n,k) = n(n-1)(n-2)…(n-k+1)= n!/(n-k)!

Combinations

A *k*-combination of elements of a set is an **unordered** selection of *k* elements from the set. Thus, a *k*-combination is simply a subset of the set with *k* elements.

Example:

- 2-combinations of the set {a,b,c}

  a b a c b c

  ↓

  a b covers two of the 2-permutations: **a b** and **b a**

Combinations

Theorem: The number of k-combinations of a set with n distinct elements, where n is a positive integer and k is an integer with $0 \leq k \leq n$ is

$$C(n,k) = \frac{n!}{(n-k)!\,k!}$$

Combinations

Example:

- We need to create a team of 5 players for the competition out of 10 team members. How many different teams is it possible to create?

Answer:

- When creating a team we do not care about the order in which players were picked. It is is important that the player is in. Because of that we need to consider unordered sets of people.

- C(10,5) = 10!/(10-5)!5! = (10.9.8.7.6) / (5 4 3 2 1)
  = 2.3.2.7.3= 6.14.3= 6.42= **252**

Binomial coefficients

- The number of k-combinations out of n elements C(n,k) is often denoted as:

$$\binom{n}{k}$$

and reads **n choose k**. The number is also called **a binomial coefficient.**

- Binomial coefficients occur as coefficients in the expansion of powers of binomial expressions such as

$$(a+b)^n$$

- **Definition:** a binomial expression is the sum of two terms *(a+b).*

Binomial coefficients

Example:

- Expansion of the binomial expression $(a+b)^3$.

$$(a+b)^3 =$$
$$(a+b)(a+b)(a+b) =$$
$$(a^2+2ab+b^2)(a+b) =$$
$$a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 =$$
$$1a^3+3a^2b+3ab^2+1b^3$$

1 **3** **3** **1** ← Binomial coefficients

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Binomial coefficients

Binomial theorem: Let a and b be variables and n be a nonnegative integer. Then:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} b^n$$

Binomial coefficients

Corrolary: Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Binomial coefficients

Corrolary:

- Let n be a nonnegative integer. Then:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

Proof:

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = \sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i} = ((-1)+1)^n = 0^n = 0$$

Binomial coefficients

Example:

- Show that $\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$

- **Answer:**

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = \sum_{i=0}^{n} \binom{n}{i} (2)^i 1^{n-i} = (2+1)^n = 3^n$$

Binomial coefficients

We have binomial coefficients for expressions with the power n.

Question: Are binomial coefficients for powers of n-1 or n+1 in any way related to coefficients for n ?

- **The answer is yes.**

Theorem:

- Let n and k be two positive integers with . Then it holds:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

	powers
1	
1 1	
1 2 1	2
1 3 3 1	3
	4

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

powers

1	
1 1	
1 2 1	**2**
1 3 3 1	**3**
1 4 6 4 1	**4**
	5

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

powers

1	
1 1	
1 2 1	**2**
1 3 3 1	**3**
1 4 6 4 1	**4**
1 5 10 10 5 1	**5**
	6

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

powers

1	
1 1	
1 2 1	**2**
1 3 3 1	**3**
1 4 6 4 1	**4**
1 5 10 10 5 1	**5**
1 6 15 20 15 6 1	**6**

Pascal traingle

Drawing the binomial coefficients for different powers in increasing order gives a Pascal triangle:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

powers

1	
1 1	
1 2 1	**2**
1 3 3 1	**3**
1 4 6 4 1	**4**
1 5 10 10 5 1	**5**
1 6 15 20 15 6 1	**6**
…	**7**

Permutations with repetitions

Assume we want count different ordered collections of objects such that we can pick the object from a set multiple times: that is, an object can be on the position 1, 2, 3, etc

Example:

- 26 letters of alphabet. How many different strings of length k are there?

Answer:

- 26^k

Permutations with repetitions

Assume we want count different ordered collections of objects such that we can pick the object from a set multiple times: that is, an object can be on the position 1, 2, 3, etc

Example:

- 26 letters of alphabet. How many different strings of length k are there?

Answer:

- 26^k

Theorem: The number of k-permutations of a set of n objects with repetitions is $\mathbf{n^k}$.

Combinations with repetitions

Example:

- Pick four pieces of a fruit from three bowls with apples, pears and oranges. How many possible combinations are there? List /count all of them?

Answer:

- **Star and bar approach**
- Apples Pears Oranges
- 3 bowls separated by | |
- Choice 2 apples and 2 pears represented as: ** | ** |
- Choice of 1 apple and 3 oranges: * | | ***

Combinations with repetitions

Example:

- Pick four pieces of a fruit from three bowls with apples, pears and oranges. How many possible combinations are there? List /count all of them?

Answer:

- **Star and bar approach**
- Apples Pears Oranges
- 3 bowls separated by | |
- Choice 2 apples and 2 pears represented as: ** | ** |
- Choice of 1 apple and 3 oranges: * | | ***
- **Count:** How many different ways of arranging (3-1)=2 bars and 4 stars are there?
- Total number of positions: 2+4=6
- **Count:** the number of ways to select the positions of 4 stars.

Combinations with repetitions

- **Theorem:** The number of ways to pick n elements from k different groups is:

$$\binom{n-1+k}{n}$$

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Combinations with repetitions

- **Theorem:** The number of ways to pick n elements from k different groups is:

$$\binom{n-1+k}{n}$$

- (n+k-1) positions
- n- stars
- **Count:** the number of ways to select the positions of 4 stars.

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Probabilities

Probability

Discrete probability theory

- Dates back to 17th century.
- It was used to compute the **odds** of seeing some outcomes: e.g. in games, races etc.
- Odds are related to counting when the outcomes are equally likely.

Example: Coin flip

- Assume 2 outcomes (head and tail) and each of them is equally likely
- Odds: 50%, 50%
- the probability of seeing:
    - a head is 0.5
    - a tail is 0.5

Probability

Probability is related to relative **counts** of target outcomes with respect to all outcomes

P = Number of target outcomes / Total number of outcomes

Example: Coin flip

- the probability of seeing:
  - a head is 0.5
  - a tail is 0.5
- Probability of all (disjoint) events is 1
- **P(any outcome) = P(head) + P(tail) =1**

Probability

Example: roll of a dice

- 6 different outcomes. Each of them is equally likely
- Probability of each outcome is:
- 1/6
- How did we get the number?

Probability

Example: roll of a dice

- 6 different outcomes. Each of them is equally likely
- Probability of each outcome is:
- 1/6
- How did we get the number?
- Assuming each outcome is equally likely and we have 6 outcomes, then the probability of an outcome is:

  1/6

Probability of aggregate outcomes

Example: roll of a dice

- Roll of the dice is odd or even. All outcomes are equally likely.
- Probability = number of outcomes when odd/ total number of outcomes.

Solution 1: all outcomes are equally likely and = 1/6

Odd numbers: 1,3,5 Even numbers: 2,4,6

- P(odd) = 1/6+1/6+1/6 = 3/6 = 1/2
- P(even) = 3/6 = 1/2

Solution 2:

Odd numbers are equally likely as even numbers (2 outcomes)

P(odd) =1/2 and P(even) = 1/2

Probabilities

- **Experiment:** a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)
- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
  - **P(Event)= |E| / |S|**

- The cardinality of the subset divided by the cardinality of the sample space.

Probabilities

Example 1:

- A box with 4 red balls and 6 blue balls. What is the probability that we pull the blue ball out.

- **P(E) =6/10 =0.6**

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
- (1,6) (2,6) …(6,1),…(6,6) total: 36

- Outcomes leading to
- (1,6) (2,5) …(6,1) total: 6
- P(sum=7)= 6/36 = 1/6

Probabilities

More complex:

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes:
  - C(40,6) = …
- Probability of winning: ?
  - P(E) =1/C(40,6)= 34! 6! / 40! = 1/3,838,380