CS 441 Discrete Mathematics for CS
Lecture 14

Integers: applications, base conversions.

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Modular arithmetic in CS

Modular arithmetic and congruencies are used in CS:

- **Pseudorandom number generators**
- **Hash functions**
- **Cryptology**

Pseudorandom number generators

- **Some problems we want to program need to simulate a random choice.**
- **Examples: flip of a coin, roll of a dice**

We need a way to generate random outcomes

Basic problem:

- assume outcomes: 0, 1, .. N
- generate the random sequences of outcomes

- Pseudorandom number generators let us generate sequences that look random
- **Next:** linear congruential method

Pseudorandom number generators

Linear congruential method

- We choose 4 numbers:
  - the modulus m,
  - multiplier a,
  - increment c, and
  - seed x_0,

  such that 2 =< a < m, 0 =< c < m, 0 =< x_0 < m.

- We generate a sequence of numbers $x_1, x_2\ x_3 \dots x_n \dots$ such that 0 =< x_n < m for all *n* by successively using the congruence:
  - $\mathbf{x_{n+1} = (a.x_n + c) \bmod m}$

Pseudorandom number generators

Linear congruential method:

- $x_{n+1} = (a.x_n + c) \bmod m$

Example:

- Assume : m=9,a=7,c=4, $x_0 = 3$

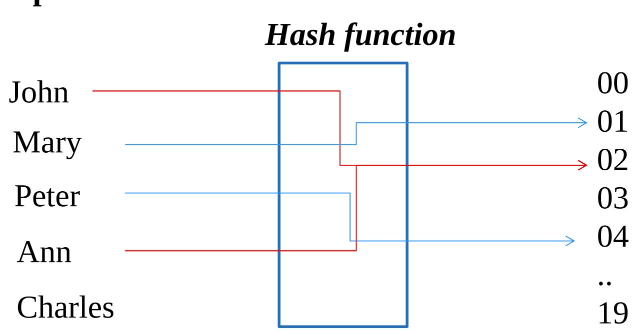
- x_1= 7*3+4 mod 9=25 mod 9 =7
- x_2 = 53 mod 9 = 8
- x_3 = 60 mod 9 = 6
- x_4= 46 mod 9 =1
- x_5 = 11 mod 9 =2
- x_6 = 18 mod 9 =0
- ....

Hash functions

A ***hash function*** is an algorithm that maps data of arbitrary length to data of a fixed length.

The values returned by a hash function are called **hash values** or **hash codes.**

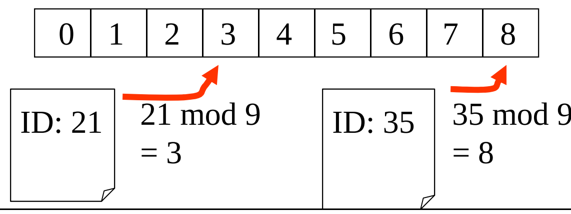
Example:



Hash functions

- **Problem:** Given a large collection of records, how can we store and find a record quickly?
- **Solution:** Use a hash function calculate the location of the record based on the record's ID.
- **Example:** A common hash function is
  - $h(k) = k \textbf{ mod } n$,

  where n is the number of available storage locations.



Hash function

An example of a hash function that maps integers (including very large ones) to a subset of integers 0, 1, .. m-1 is:

h(k) = k mod m

Example: Assume we have a database of employes, each with a unique ID – a social security number that consists of 8 digits. We want to store the records in a smaller table with m entries. Using h(k) function we can map a social secutity number in the database of employes to indexes in the table.

Assume: h(k) = k mod 111

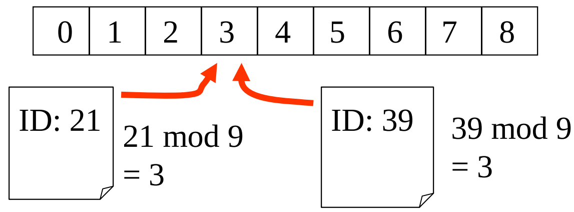
Then:

h(064212848) = 064212848 mod 111 = 14

h(037149212) = 037149212 mod 111 = 65

Hash functions

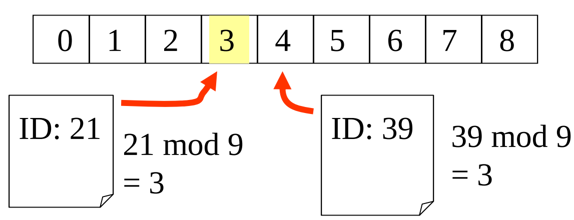
- **Problem:** two documents mapped to the same location



Hash functions

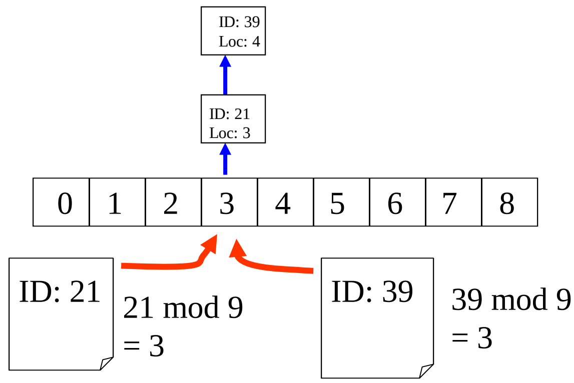
- **Solution 1:** move the next available location
  - Method is represented by a sequence of hash functions to try

$$h_0(k) = k \textbf{ mod } n$$
$$h_1(k) = (k+1) \textbf{ mod } n$$
$$\ldots$$
$$h_m(k) = (k+m) \textbf{ mod } n$$



Hash functions

- **Solution 2:** remember the exact location in a secondary structure that is searched sequentially



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Cryptology

Encryption of messages.

- **Ceasar cipher:**
- Shift letters in the message by 3, last three letters mapped to the first 3 letters, e.g. A is shifted to D, X is shifted to A

How to represent the idea of a shift by 3?

- There are 26 letters in the alphabet. Assign each of them a number from 0,1, 2, 3, .. 25 according to the alphabetical order.

A B C D E F G H I J K L M N O P Q R S T U Y V X W Z

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

- The encryption of the letter with an index p is represented as:
    - $f(p) = (p + 3) \bmod 26$

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Cryptology

Encryption of messages using a shift by 3.

- The encryption of the letter with an index p is represented as:
  - f(p) = (p + 3) mod 26

Coding of letters:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	Y	V	X	W	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

- **Encrypt message:**
  - **I LIKE DISCRETE MATH**
  -

Cryptology

Encryption of messages using a shift by 3.

- The encryption of the letter with an index p is represented as:
  - f(p) = (p + 3) mod 26

Coding of letters:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	Y	V	X	W	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

- **Encrypt message:**
  - **I LIKE DISCRETE MATH**
  - **L 0LNH GLYFUHVH PDVK.**

Cryptology

How to decode the message ?

- The encryption of the letter with an index p is represented as:
  - $f(p) = (p + 3) \bmod 26$

Coding of letters:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	Y	V	X	W	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

- **What method would you use to decode the message:**
  - $\mathbf{f^{-1}(p) = (p-3) \bmod 26}$

Representations of Integers

- In the modern world, we use *decimal,* or *base* 10, *notation* to represent integers. For example when we write 965, we mean $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$.
- We can represent numbers using any base b, where b is a positive integer greater than 1.
- The bases $b = 2$ (*binary*), $b = 8$ (*octal*) , and $b= 16$ (*hexadecimal*) are important for computing and communications
- The ancient Mayans used base 20 and the ancient Babylonians used base 60.

Base b Representations

- We can use positive integer b greater than 1 as a base

Theorem 1: Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form:

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots. + a_1 b + a_0$$

where k is a nonnegative integer, $a_0, a_1, \ldots.\ a_k$ are nonnegative integers less than b, and $a_k \neq 0$. The a_j, $j = 0, \ldots, k$ are called the base-b digits of the representation.

- The representation of n given in **Theorem 1** is called the ***base b expansion of n*** and is denoted by $(a_k a_{k-1} \ldots. a_1 a_0)_b$.
- We usually omit the subscript 10 for base 10 expansions.

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Binary Expansions

Most computers represent integers and do arithmetic with binary (base 2) expansions of integers. In these expansions, the only digits used are 0 and 1.

Example: What is the decimal expansion of the integer that has $(1\ 0101\ 1111)_2$ as its binary expansion?

Solution:

$(1\ 0101\ 1111)_2 = 1{\cdot}2^8 + 0{\cdot}2^7 + 1{\cdot}2^6 + 0{\cdot}2^5 + 1{\cdot}2^4 + 1{\cdot}2^3 + 1{\cdot}2^2 + 1{\cdot}2^1 + 1{\cdot}2^0 = 351.$

Example: What is the decimal expansion of the integer that has $(11011)_2$ as its binary expansion?

Solution: $(11011)_2 = 1{\cdot}2^4 + 1{\cdot}2^3 + 0{\cdot}2^2 + 1{\cdot}2^1 + 1{\cdot}2^0 = 27$

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Octal Expansions

The octal expansion (base 8) uses the digits $\{0,1,2,3,4,5,6,7\}$.

Example: What is the decimal expansion of the number with octal expansion $(7016)_8$?

Solution: $7 \cdot 8^3 + 0 \cdot 8^2 + 1 \cdot 8^1 + 6 \cdot 8^0 = 3598$

Example: What is the decimal expansion of the number with octal expansion $(111)_8$?

Solution: $1 \cdot 8^2 + 1 \cdot 8^1 + 1 \cdot 8^0 = 64 + 8 + 1 = 73$

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Hexadecimal Expansions

- The hexadecimal expansion uses 16 digits: $\{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}$.
  - The letters A through F represent the decimal numbers 10 through 15.

Example: What is the decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

Solution:

$2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16^1 + 11 \cdot 16^0 = 175627$

Example: What is the decimal expansion of the number with hexadecimal expansion $(E5)_{16}$?

Solution: $14 \cdot 16^1 + 5 \cdot 16^0 = 224 + 5 = 229$

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Base Conversion

To construct the base b expansion of an integer n:

- Divide n by b to obtain a quotient and remainder.

  $n = bq_0 + a_0 \quad 0 \le a_0 \le b$
- The remainder, a_0, is the rightmost digit in the base b expansion of n. Next, divide q_0 by b.

  $q_0 = bq_1 + a_1 \quad 0 \le a_1 \le b$
- The remainder, a_1, is the second digit from the right in the base b expansion of n.
- Continue by successively dividing the quotients by b, obtaining the additional base b digits as the remainder. The process terminates when the quotient is 0.

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Base Conversion

Example: Find the octal expansion of $(12345)_{10}$

Solution: Successively dividing by 8 gives:

- $12345 = 8 \cdot 1543 + 1$
- $1543 = 8 \cdot 192 + 7$
- $192 = 8 \cdot 24 + 0$
- $24 = 8 \cdot 3 + 0$
- $3 = 8 \cdot 0 + 3$

The remainders are the digits from right to left yielding $(30071)_8$.

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Mathematical induction & Recursion

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Proofs

Basic proof methods:

- Direct, Indirect, Contradiction, By Cases, Equivalences

Proof of quantified statements:

- **There exists x with some property P(x).**
  - It is sufficient to find one element for which the property holds.
- **For all x some property P(x) holds.**
  - Proofs of 'For all x some property P(x) holds' must cover all x and can be harder.
- **Mathematical induction** is a technique that can be applied to prove the universal statements for sets of positive integers or their associated sequences.

Mathematical induction

- Used to prove statements of the form $\forall x\ P(x)$ where $x \in Z^+$

Mathematical induction proofs consists of two steps:

1) **Basis:** The proposition P(1) is true.
2) **Inductive Step:** The implication

   $P(n) \rightarrow P(n+1)$, is true for all positive n.

- Therefore we conclude $\forall x\ P(x)$.

- **Based on the well-ordering property:** Every nonempty set of nonnegative integers has a **least element**.

Mathematical induction

Example: Prove the sum of first n odd integers is n^2 .

i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

Proof:

- What is P(n)? P(n): $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$

Basis Step Show P(1) is true

- Trivial: $1 = 1^2$

Inductive Step Show if P(n) is true then P(n+1) is true for all n.

- Suppose P(n) is true, that is $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$
- Show P(n+1): $1 + 3 + 5 + 7 + ... + (2n - 1) + (2n + 1) = (n+1)^2$ follows:
- $\underbrace{1 + 3 + 5 + 7 + ... + (2n - 1)}_{n^2} + (2n + 1) =$

  $n^2 + (2n+1) = (n+1)^2$