CS 441 Discrete Mathematics for CS
Lecture 11

Countable and uncountable sets. Matrices.

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Arithmetic series

Definition: The sum of the terms of the **arithmetic progression** a, a+d,a+2d, …, a+nd is called an **arithmetic series**.

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n} (a + jd) = na + d\sum_{j=1}^{n} j = na + d\,\frac{n(n+1)}{2}$$

Geometric series

Definition: The sum of the terms of **a geometric progression** a, ar, ar^2, ..., ar^k is called **a geometric series**.

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$S = \sum_{j=0}^{n} (ar^j) = a \sum_{j=0}^{n} r^j = a \left[\frac{r^{n+1} - 1}{r - 1} \right]$$

Infinite geometric series

- **Infinite geometric series can be computed in the closed form for x<1**
- **How?**

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \sum_{n=0}^{k} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = -\frac{1}{x - 1} = \frac{1}{1 - x}$$

- **Thus:**

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

Cardinality

Recall: The cardinality of a finite set is defined by the number of elements in the set.

Definition: The sets A and B have **the same cardinality** if there is a one-to-one correspondence between elements in A and B. In other words if there is a bijection from A to B. Recall bijection is one-to-one and onto.

Example: Assume $A = \{a,b,c\}$ and $B = \{\alpha,\beta,\gamma\}$ and function f defined as:

- $a \to \alpha$
- $b \to \beta$
- $c \to \gamma$

F defines a bijection. Therefore A and B have the same cardinality, i.e. $|A| = |B| = 3$.

Cardinality

Definition: A set that is either finite or has the same cardinality as the set of positive integers Z^+ is called **countable.** A set that is not countable is called **uncountable.**

Why these are called countable?

- **The elements of the set can be enumerated and listed.**

Countable sets

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?

Countable sets

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?
- Using the definition: Is there a bijective function f: $Z^+ \rightarrow A$

Z+ = {1, 2, 3, 4, …}

Countable sets

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?
- Using the definition: Is there a bijective function f: $Z^+ \rightarrow A$

Z+ = {1, 2, 3, 4, …}

- Define a function f: $x \rightarrow 2x - 2$ (an arithmetic progression)
  - $1 \rightarrow 2(1)-2 = 0$
  - $2 \rightarrow 2(2)-2 = 2$
  - $3 \rightarrow 2(3)-2 = 4$ ...

Countable sets

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?
- Using the definition: Is there a bijective function f: $Z^+ \rightarrow A$

Z+ = {1, 2, 3, 4, …}

- Define a function f: $x \rightarrow 2x - 2$ (an arithmetic progression)
  - $1 \rightarrow 2(1)-2 = 0$
  - $2 \rightarrow 2(2)-2 = 2$
  - $3 \rightarrow 2(3)-2 = 4$ ...
- one-to-one (why?)

Countable sets

Example:

- Assume A = {0, 2, 4, 6, ... } set of even numbers. Is it countable?
- Using the definition: Is there a bijective function f: $Z^+ \rightarrow A$

Z+ = {1, 2, 3, 4, …}

- Define a function f: $x \rightarrow 2x - 2$ (an arithmetic progression)
  - $1 \rightarrow 2(1)-2 = 0$
  - $2 \rightarrow 2(2)-2 = 2$
  - $3 \rightarrow 2(3)-2 = 4$ ...
- one-to-one (why?) 2x-2 = 2y-2 => 2x = 2y =>x = y.
- onto (why?) $\forall a \in A$, (a+2) / 2 is the pre-image in Z^+.
- Therefore | A | = | Z^+ |.

Countable sets

Theorem:

- **The set of integers Z is countable**.

Solution:

Can list a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots$

Or can define a bijection from Z^+ to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

Countable sets

Definition:

- A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.

Theorem:

- **The positive rational numbers are countable.**

Solution:

The positive rational numbers are countable since they can be arranged in a sequence:

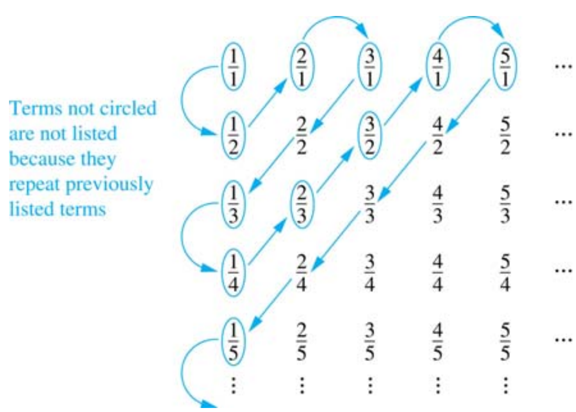
$$r_1, r_2, r_3, \ldots$$

Countable sets

Theorem:

- **The positive rational numbers are countable.**

First row $q = 1$.
Second row $q = 2$.
etc.



Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

Cardinality

Theorem: The set of real numbers (R) is an uncountable set.

Proof by a contradiction.

1) Assume that the real numbers are countable.
2) Then every subset of the reals is countable, in particular, the interval from 0 to 1 is countable. This implies the elements of this set can be listed say r1, r2, r3, ... where

- $r1 = 0.d_{11}d_{12}d_{13}d_{14} \ldots$
- $r2 = 0.d_{21}d_{22}d_{23}d_{24} \ldots$
- $r3 = 0.d_{31}d_{32}d_{33}d_{34} \ldots\ldots$
- where the $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$.

Real numbers are uncountable

Proof cont.

3) Want to show that not all reals in the interval between 0 and 1 are in this list.

- Form a new number called
  - $r = 0.d_1d_2d_3d_4 \ldots$ where

$$d_i = \begin{cases} 2, \text{ if } d_{ii} \neq 2 \\ 3 \text{ if } d_{ii} = 2 \end{cases}$$

- Example: suppose

r1 = 0.**7**5243...	d1 = 2
r2 = 0.5**2**4310...	d2 = 3
r3 = 0.13**1**257...	d3 = 2
r4 = 0.936**3**633...	d4 = 2
...	...
rt = 0.23222**2**22...	dt = 3

Real numbers are uncountable

- $r = 0.d_1d_2d_3d_4 \ldots$ where

$$d_i = \begin{cases} 2, \text{ if } d_{ii} \neq 2 \\ 3 \text{ if } d_{ii} = 2 \end{cases}$$

- **Claim:** r is different than each member in the list.
- Is each expansion unique? Yes, if we exclude an infinite string of 9s.
-
- Example: $.0285\bar{0} = .0284\bar{9}$
- Therefore r and r_i differ in the i-th decimal place for all i.

Matrices

Matrices

Definitions:

- **A matrix** is a rectangular array of numbers.
- A matrix with ***m*** **rows and** ***n*** **columns** is called an m x n matrix.

Note: The plural of matrix is *matrices*.

Matrices

Definitions:

- **A matrix** is a rectangular array of numbers.
- A matrix with ***m*** **rows and** ***n*** **columns** is called an m x n matrix.

Note: The plural of matrix is *matrices*.

Definitions:

- A matrix with the same number of rows as columns is called a *square matrix*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Matrices

- Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The ith row of **A** is the 1 x n matrix $[a_{i1}, a_{i2},\ldots,a_{in}]$. The jth column of **A** is the m x 1 *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$$

- The (i,j)th *element* or *entry* of **A** is the element a_{ij}. We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (i,j)th element equal to a_{ij}.

Matrix addition

Defintion:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be m x n matrices. The sum of **A** and **B**, denoted by **A** + **B**, is the m x n matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note: matrices of different sizes can not be added.

Matrix multiplication

Definition:

- Let **A** be an m x k matrix and **B** be a k x n matrix. The *product* of **A** and **B**, denoted by **AB**, is the m x n matrix that has its (i,j)th element equal to the sum of the products of the corresponding elments from the ith row of **A** and the jth column of **B**. In other words, if

  AB = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{jk}b_{kj}$.

Example:

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

- The product is not defined when the number of columns in the first matrix is not equal to the number of rows in the second matrix

Matrix multiplication

The Product of **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ . & . & & . \\ . & . & & . \\ a_{i1} & a_{i2} & \ldots & a_{ik} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \ldots & a_{mk} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & a_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\ . & . & & . & & \\ . & . & & . & & \\ b_{k1} & b_{k2} & \ldots & b_{kj} & \ldots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ . & . & & . \\ . & . & c_{ij} & . \\ . & . & & . \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Matrix multiplication

- **Example**:

$$\begin{vmatrix} 2 & 4 & 3 \\ 1 & 1 & 2 \\ 2 & 4 & 6 \end{vmatrix} * \begin{vmatrix} 1 & 4 & 2 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{vmatrix} = \begin{vmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{vmatrix}$$

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$\mathbf{AB} \neq \mathbf{BA}$

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB: ?**

CS 441 Discrete mathematics for CS M. Hauskrecht

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB:**

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

CS 441 Discrete mathematics for CS M. Hauskrecht

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB:** **BA: ?**

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Matrix multiplication

Properties of matrix multiplication:

- Does **AB** = **BA**?

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- **AB:** **BA:**

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

- **Conclusion: AB ≠ BA**

Matrices

Definition:

- **The identity matrix (of order n)** is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

Properties:

- Assume A is an $m \times n$ matrix. Then:

  $\mathbf{AI}_n = \mathbf{A}$ and $\mathbf{I}_m\mathbf{A} = \mathbf{A}$

- Assume A is an $n \times n$ matrix. Then: $\mathbf{A}^0 = \mathbf{I}_n$

Matrices

Definition: **Powers of square matrices**

- When A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{r}$$

Matrix transpose

Definition:

- Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of **A**, denoted by $\mathbf{A}^T$,is the $n \times m$ matrix obtained by interchanging the rows and columns of **A**.

If $\mathbf{A}^T = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1,2,\ldots,n$ and $j = 1,2, \ldots,m$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Matrix inverse

Definition:

- Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The **inverse** of **A**, denoted by $\mathbf{A}^{-1}$, is the $n \times m$ matrix such that $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$

- Note: the inverse of the matrix **A** may not exist.