CS 441 Discrete Mathematics for CS
Lecture 10

Sequences and summations

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Sequences

Definition: A **sequence is a function** from a subset of the set of integers (typically the set {0,1,2,...} or the set {1,2,3,...} to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Notation: $\{a_n\}$ is used to represent the sequence (note {} is the same notation used for sets, so be careful). $\{a_n\}$ represents the ordered list $a_1, a_2, a_3, \ldots$.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \ldots \end{array}$$

$\{a_n\}$

Sequences

Examples:

- (1) $a_n = n^2$, where $n = 1,2,3...$
  - What are the elements of the sequence?

    1, 4, 9, 16, 25, ...
- (2) $a_n = (-1)^n$, where n=0,1,2,3,...
  - Elements of the sequence?

    1, -1, 1, -1, 1, ...
- 3) $a_n = 2^n$, where n=0,1,2,3,...
  - Elements of the sequence?

    1, 2, 4, 8, 16, 32, ...

Arithmetic progression

Definition: An **arithmetic progression** is a sequence of the form

a, a+d,a+2d, …, a+nd

where a is the *initial term* and d is *common difference*, such that both belong to R.

Example:

- $s_n = -1+4n$ for n=0,1,2,3, …
- members: -1, 3, 7, 11, …

Geometric progression

Definition A **geometric progression** is a sequence of the form:

$a, ar, ar^2, ..., ar^k,$

where a is the *initial term*, and r is the *common ratio*. Both a and r belong to R.

Example:

- $a_n = (\frac{1}{2})^n$ for $n = 0,1,2,3, \dots$

  members: 1,½, ¼, 1/8, …..

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

Example:

- Assume the sequence: 1,3,5,7,9, ….
- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.

  1, 1+2=3, 3+2=5, 5+2=7
- What type of progression this suggest?

Sequences

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Example:

- Assume the sequence: 1,3,5,7,9, ….
- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.
- 1, 1+2=3, 3+2=5, 5+2=7
- It suggests **an arithmetic progression**: a+nd

  with a=1 and d=2

  - $a_n=1+2n$

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

Example 2:

- Assume the sequence: 1, 1/3, 1/9, 1/27, …
- What is the sequence?
- The denominators are powers of 3.

  1, 1/3= 1/3, (1/3)/3=1/(3*3)=1/9, (1/9)/3=1/27
- This suggests a **geometric progression**: ar^k

  with a=1 and r=1/3

  - $(1/3)^n$

Recursively defined sequences

- The n-th element of the sequence $\{a_n\}$ is defined recursively in terms of the previous elements of the sequence and the initial elements of the sequence.

Example :

- $a_n = a_{n-1} + 2$ assuming $a_0 = 1$;
- $a_0 = 1$;
- $a_1 = 3$;
- $a_2 = 5$;
- $a_3 = 7$;
- Can you write a_n non-recursively using n?
- $a_n = 1 + 2n$

Fibonacci sequence

- Recursively defined sequence, where
- $f_0 = 0$;
- $f_1 = 1$;
- $f_n = f_{n-1} + f_{n-2}$ for n = 2,3, …

- $f_2 = 1$
- $f_3 = 2$
- $f_4 = 3$
- $f_5 = 5$

Summations

Summation of the terms of a sequence:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

The variable j is referred to as the index of summation.

- m is the *lower limit* and
- n is the *upper limit* of the summation.

Summations

Example:

- **1) Sum the first 7 terms of {n^2} where n=1,2,3, ...**
- $$\sum_{j=1}^{7} a_j = \sum_{j=1}^{7} j^2 = 1 + 4 + 16 + 25 + 36 + 49 = 140$$

- **2) What is the value of**

$$\sum_{k=4}^{8} a_j = \sum_{k=4}^{8} (-1)^j = 1 + (-1) + 1 + (-1) + 1 = 1$$

Arithmetic series

Definition: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is called an **arithmetic series**.

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n} (a + jd) = na + d\sum_{j=1}^{n} j = na + d\frac{n(n+1)}{2}$$

- **Why?**

Arithmetic series

Theorem: The sum of the terms of the arithmetic progression a, a+d,a+2d, …, a+nd is

$$S = \sum_{j=1}^{n} (a + jd) = na + d\sum_{j=1}^{n} j = na + d\frac{n(n+1)}{2}$$

Proof:

$$S = \sum_{j=1}^{n} (a + jd) = \sum_{j=1}^{n} a + \sum_{j=1}^{n} jd = na + d\sum_{j=1}^{n} j$$

$$\sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + ... + (n-2) + (n-1) + n$$

Arithmetic series

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Proof:

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$$\sum_{j=1}^{n} j = 1 + 2 + 3 + 4 + ... + (n-2) + (n-1) + n$$

$$\underbrace{n+1 \qquad n+1 \quad \ldots \quad n+1}_{\frac{n}{2}*(n+1)}$$

Arithmetic series

Example:

$$S = \sum_{j=1}^{5}(2 + j3) =$$

$$= \sum_{j=1}^{5} 2 + \sum_{j=1}^{5} j3 =$$

$$= 2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j =$$

$$= 2*5 + 3\sum_{j=1}^{5} j =$$

$$= 10 + 3\frac{(5+1)}{2}*5 =$$

$$= 10 + 45 = 55$$

Arithmetic series

Example 2:

$$S = \sum_{j=3}^{5} (2 + j3) =$$

$$= \left[\sum_{j=1}^{5} (2 + j3) \right] - \left[\sum_{j=1}^{2} (2 + j3) \right]$$

Trick

$$= \left[2\sum_{j=1}^{5} 1 + 3\sum_{j=1}^{5} j \right] - \left[2\sum_{j=1}^{2} 1 + 3\sum_{j=1}^{2} j \right]$$

$$= 55 - 13 = 42$$

Double summations

Example:

$$S = \sum_{i=1}^{4} \sum_{j=1}^{2} (2i - j) =$$

$$= \sum_{i=1}^{4} \left[\sum_{j=1}^{2} 2i - \sum_{j=1}^{2} j \right] =$$

$$= \sum_{i=1}^{4} \left[2i \sum_{j=1}^{2} 1 - \sum_{j=1}^{2} j \right] =$$

$$= \sum_{i=1}^{4} \left[2i * 2 - \sum_{j=1}^{2} j \right] =$$

$$= \sum_{i=1}^{4} [2i * 2 - 3] =$$

$$= \sum_{i=1}^{4} 4i - \sum_{i=1}^{4} 3 =$$

$$= 4\sum_{i=1}^{4} i - 3\sum_{i=1}^{4} 1 = \quad 4*10 - 3*4 = 28$$

Geometric series

Definition: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^k is called **a geometric series**.

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$S = \sum_{j=0}^{n} (ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Geometric series

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, ..., ar^n is

$$S = \sum_{j=0}^{n} (ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Proof:

$$S = \sum_{j=0}^{n} ar^j = a + ar + ar^2 + ar^3 + \ldots + ar^n$$

- multiply S by r

$$rS = r\sum_{j=0}^{n} ar^j = ar + ar^2 + ar^3 + \ldots + ar^{n+1}$$

- Substract $rS - S = \left[ar + ar^2 + ar^3 + \ldots + ar^{n+1}\right] - \left[a + ar + ar^2 .. + ar^n\right]$

$$= ar^{n+1} - a$$

$$\Rightarrow \quad S = \frac{ar^{n+1}-a}{r-1} = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

Geometric series

Example:

$$S = \sum_{j=0}^{3} 2(5)^j =$$

General formula:

$$S = \sum_{j=0}^{n} (ar^j) = a\sum_{j=0}^{n} r^j = a\left[\frac{r^{n+1}-1}{r-1}\right]$$

$$S = \sum_{j=0}^{3} 2(5)^j = 2*\frac{5^4-1}{5-1} =$$

$$= 2*\frac{625-1}{4} = 2*\frac{624}{4} = 2*156 = 312$$

Infinite geometric series

- **Infinite geometric series can be computed in the closed form for x<1**
- **How?**

$$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \sum_{n=0}^{k} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1} = -\frac{1}{x-1} = \frac{1}{1-x}$$

- **Thus:**

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$