CS441 – Discrete Structures for Computer Science
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Problems from Chapter 4.1, 4.2, 4.3 and 5.1 (6th Edition) Total Points: 110

Chapter 4.1

4. Let P(n) be the proposition $1^3 + 2^3 + \cdots + n^3 = (n(n + 1)/2)^2$ whenever n is a positive integer.

Basis step: We verify P(1), namely that 13 = [1 · (1 + 1)/2]2 which is indeed true. Next,
Inductive step: We assume that P(k) is true (the inductive hypothesis), and try to derive P(k + 1). Now, P(k + 1) is the formula
$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = ((k + 1)(k + 2)/2)^2$

Observe that all but the last term of the left-hand side (lhs) is exactly the lhs of P(k), so by the inductive hypothesis, it equals $(k(k + 1)/2)^2$. Thus we have,
$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 =$
$$((k + 1) k/2)^2 + (k+1)^3 = (k+1)^2((k^2/4)+k+1) = ((k + 1)(k + 2)/2)^2$$

7. Let P(n) be the proposition $3+3 \cdot 5+3 \cdot 5^2 \cdots +3 \cdot 5^n = 3(5^{n+1}-1)/4$. To prove that P(n) is true for all nonnegative integers n, we proceed by mathematical induction.

Basis step: We verify P(0), namely that 3 = 3(5−1)/4 which is indeed true. Next,
Inductive step: We assume that P(k) is true (the inductive hypothesis), and try to derive P(k + 1). Now, P(k + 1) is the formula
$3 + 3 \cdot 5 + 3 \cdot 5^2 \cdots + 3 \cdot 5^k + 3 \cdot 5^{K+1} = 3(5^{K+2} - 1)/4$
Observe that all but the last term of the left-hand side (lhs) is exactly the lhs of P(k), so by the inductive hypothesis, it equals $3(5^{k+1}-1)/4$. Thus we have,
$3 + 3 \cdot 5 + 3 \cdot 5^2 \cdots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = 3(5^{k+1} - 1)/4 + 3 \cdot 5k+1$
$= 5^{k+1} (3/4+ 3) -3/4 = 3(5^{k+2} - 1)/4$

10.
By looking at the first few sums, we guess that the sum is n/(n + 1). We prove this by induction. Let P(n) be the proposition

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

Basis step: We verify P(1), namely that 1/2 = 1/(1 · 2) which is indeed true. Next,
Inductive step: We assume that P(k) is true (the inductive hypothesis), and try to derive P(k + 1). Now, P(k + 1) is the formula

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{k\cdot(k+1)}+\frac{1}{(k+1)\cdot(k+2)}=\frac{k+1}{k+2}$$

Observe that all but the last term of the left-hand side (lhs) is exactly the lhs of P(k), so by the inductive hypothesis, it equals k/(k + 1). Thus we have,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{k\cdot(k+1)}+\frac{1}{(k+1)\cdot(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

18.

The base case is n = 2, and indeed $2! < 2^2$. Assume the inductive hypothesis. Then $(k + 1)! = (k + 1) \cdot k! < (k + 1)k^k < (k + 1)(k + 1)^k = (k + 1)^{(k+1)}$.

32.

The statement is true for the base case, n = 1, since 3|3. Suppose that $3|(k^3 + 2k)$. We must show that $3|((k + 1)^3 + 2(k + 1))$. If we expand the expression in question, we obtain $k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)$. By the inductive hypothesis, 3 divides $k^3 +2k$, and certainly 3 divides $3(k^2 +k +1)$, so 3 divides their sum, and we are done.

Chapter 4.2

4.

a) P(18) is true because we can form 18 cents with one 4-cent and two 7-cent stamps.
   P(19) is true because we can form 19 cents with three 4-cent and one 7-cent stamps.
   P(20) is true because we can form 20 cents with five 4-cent stamps.
   P(21) is true because we can form 18 cents with three 7-cent stamps.

b) The inductive hyp. Is the statement that just using 4-cent and 7-cent stamps we can form j cents postage for all j with 18≤j≤k, where we assume k ≥21.

c) In the inductive step we must show, assuming the inductive hypothesis that we can form k+1 cents postage using just 4-cent and 7-cent stamps.

d) We want to form k+1 cents of postage. Since k21, we know that P(k-3) is true, that is that we can form k-3 cents of postage. Put one more 4-cent stamp on the envelope, and we have formed k+1 cents of postage, as desired.

e) As having completed both the basis step and inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

10.

Claim: Takes exactly n-1 breaks to separate a bar into n pieces.
If n=1, this is trivially true, as one piece no breaks. Assume strong inductive hypothesis. The statement is true for breaking into k or fewer pieces
To obtain k+1 pieces and show it takes exactly k breaks.

The process starts with a break leaving two smaller pieces. Rest of the process as breaking one of these pieces into i+1 pieces and breaking the other piece into k-i pieces, for some i between 0 and k-i, inclusive.
By inductive hypothesis, it will exactly take i breaks to handle first piece and k-i-1 breaks to handle breaks to handle the second piece.
Total breaks: 1 + i + (k-i-1) = k, as needed.

Chapter 4.3

2. a) f(1)=-2f(0)=-2.3=-6, f(2)=-2f(1)=-2(-6)=12, f(3)=-2f(2)=-24, f(4)=-2f(3)=-2(-24) =48, f(5)=-2f(4)=-96.

   Similarly, f(1) ,f(2), f(3), f(4) ,f(5) for
   b) 16,55,172,523,1576
   c) 1,-3,13,141,19597
   d) 3,3,3,3,3

8. a) a_1=2 and a_{n+1}=a_n+4, n≥1 (Each term is 4 more than the term before it.)
   b) a_1=0 and a_2=2 and a_n=a_{n-2}, n≥3 (0,2,0,2,…)
   c) a_1=2 and a_n=a_{n-1}+2n (2,6,12,20,30,…)
   d) a_1=1 and a_n= a_{n-1}+2n-1 (1,4,9,16,25,…)

24.
   a) 1∈S; and if n ∈ S, then n+2 ∈ S
   b) 3∈S; and if n ∈ S, then 3n ∈ S
   c) if p(x) ∈S and n is any integer, then xp(x)+n is in S. (variable for these polynomials is the letter x, all integers are in S(this base case gives us all the constant polynomials.))

Chapter 5.1

2. By the product rule 27 * 37 = 999 offices

4. By the product rule 12 * 2 * 3 = 72 different types of shirts

8. By the product rule 26 * 25 * 24 = 15600

12. $2^0+2^1+2^2+2^3+2^4+2^5+2^6 = 2^7-1 = 127$

16. Number of possible 4-letter (lower case) strings: 26^4
    Number of possible 4-letter strings without any ‘x’ : 25^4
    Thus, number of possible 4-letter strings with one or more ‘x’ = $26^4 – 25^4$

27. 2-letters, 4 digits: By product rule: 26^2*10^4; 4-letters, 2-digits: 10^2*26^4. Thus answer is $26^2 * 10^4 + 10^2 * 26^4$

42. $2^5+2^4-2^2=44$

53. Each letter is intended to be used exactly once. Consider the case where ‘a’ does not come at the end of the string. Then there are three places to put it. After we place ‘a’, only 2 places are left to put ‘b’. There are 2 possible positions for ‘c’ and only one for ‘d’. Thus there are 3*2*2*1 = 12 allowable strings where ‘a’ does not come at the end. Now we consider the case where ‘a’ comes at the end. There are 3*2*1 = 6 ways to arrange ‘b’, ’c’, and ‘d’ in the first three positions. The answer by the sum rule is therefore 12+6 = 18.