CS 3750 Machine Learning
Lecture 5

Markov Random Fields

Milos Hauskrecht
milos@cs.pitt.edu
5329 Sennott Square

Markov random fields

- **Probabilistic models with symmetric dependences.**
  - Typically models spatially varying quantities

$$P(x) \propto \prod_{c \in cl(x)} \phi_c(x_c)$$

$\phi_c(x_c)$ - A potential function (defined over factors)

- If $\phi_c(x_c)$ is strictly positive we can rewrite the definition as:

$$P(x) = \frac{1}{Z} \exp\left(- \sum_{c \in cl(x)} E_c(x_c) \right)$$ - Energy function

- Gibbs (Boltzman) distribution

$$Z = \sum_{x \in \{x\}} \exp\left(- \sum_{c \in cl(x)} E_c(x_c) \right)$$ - A partition function

Graphical representation of MRFs

An undirected network (also called independence graph)

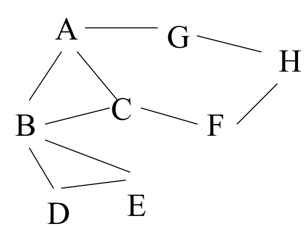
- G = (S, E)
  - S=1, 2, .. N correspond to random variables
  - $(i, j) \in E \Leftrightarrow \exists c : \{i, j\} \subset c$

    or x_i and x_j appear within the same factor c

Example:

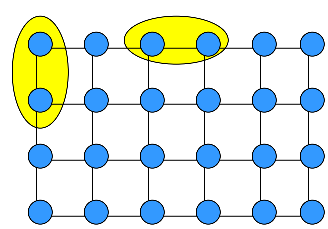
- variables A,B ..H
- Assume the full joint of MRF

$$P(A, B, \ldots H) \sim \phi_1(A, B, C)\phi_2(B, D, E)\phi_3(A, G)\phi_4(C, F)\phi_5(G, H)\phi_6(F, H)$$

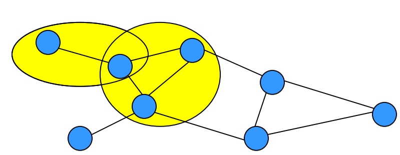


Markov random fields

- **regular lattice (Ising model)**



- **Arbitrary graph**



Markov random fields

- **Pairwise Markov property**
  - Two nodes in the network that are not directly connected can be made independent given all other nodes
- **Local Markov property**
  - A set of nodes (variables) can be made independent from the rest of nodes variables given its immediate neighbors
- **Global Markov property**
  - A vertex set A is independent of the vertex set B (A and B are disjoint) given set C if all chains in between elements in A and B intersect C

Types of Markov random fields

- **MRFs with discrete random variables**
  - Clique potentials can be defined by mapping all clique-variable instances to R
  - Example: Assume two binary variables A,B with values {a1,a2,a3} and {b1,b2} are in the same clique c. Then:

$\phi_c(A,B) \cong$

a1	b1	0.5
a1	b2	0.2
a2	b1	0.1
a2	b2	0.3
a3	b1	0.2
a3	b2	0.4

Types of Markov random fields

- **Gaussian Markov Random Field**

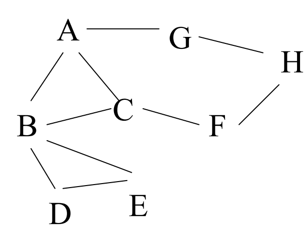
$$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

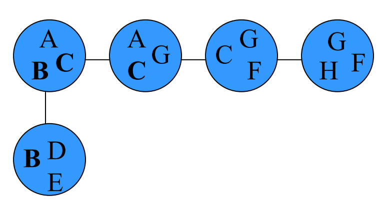
$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

- **Precision matrix** $\boldsymbol{\Sigma}^{-1}$
- **Variables in x are connected in the network only if they have a nonzero entry in the precision matrix**
  - All zero entries are not directly connected
  - Why?

Tree decomposition of the graph

- **A tree decomposition of a graph G:**
  - A tree *T* with a vertex set associated to every node.
  - For all edges $\{v,w\} \in$ G: there is a set containing both *v* and *w* in *T*.
  - For every $v \in$ G : the nodes in T that contain *v* form a connected subtree.





Tree decomposition of the graph

- **A tree decomposition of a graph G:**
  - A tree *T* with a vertex set associated to every node.
  - For all edges {*v*,*w*}∈G: there is a set containing both *v* and *w* in *T*.
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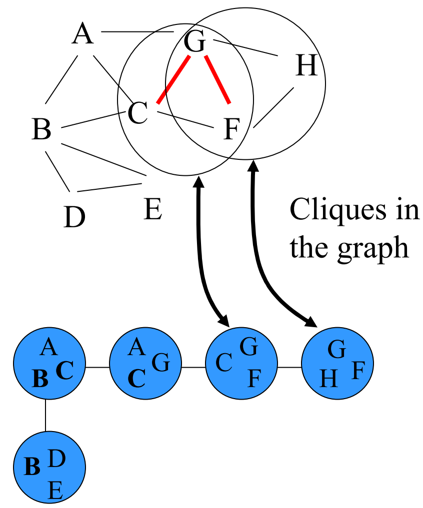
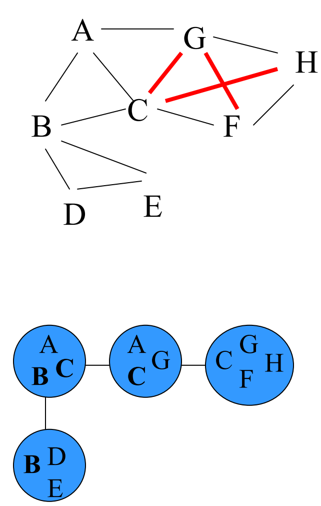


Tree decomposition of the graph

- **Another tree decomposition of a graph G:**
  - A tree *T* with a vertex set associated to every node.
  - For all edges {*v*,*w*}∈G: there is a set containing both *v* and *w* in *T*.
  - For every *v* ∈G : the nodes in T that contain *v* form a connected subtree.

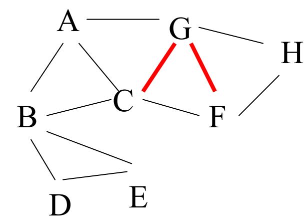


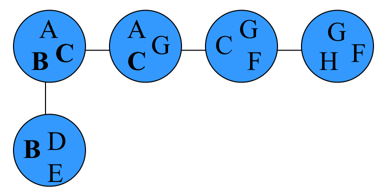
Treewidth of the graph

- **Width** of the tree decomposition:

  $\max_{i \in I} |X_i| - 1$

- **Treewidth** of a graph G: tw(G)= minimum width over all tree decompositions of G.

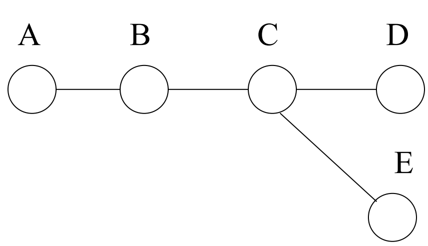




Trees

Why do we like trees?

- Inference in trees structures can be done in time **linear in the number of nodes**



Clique tree

- Clique tree = a tree decomposition of the graph
- Can be constructed:
  - from the induced graph
    Built by running the variable elimination procedure
  - from the chordal graph
    Built by running the triangulation algorithm

- We have precompiled the clique tree.
- So how to take advantage of the clique tree to perform inferences?

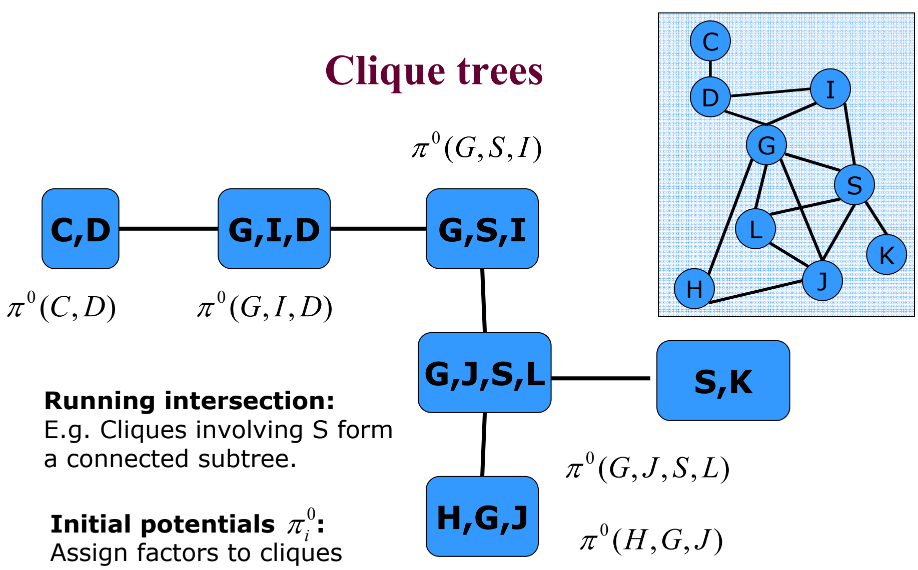
VE on the Clique tree

- Variable Elimination on the clique tree
  - works on *factors*
- Makes factor a data structure
  - Sends and receives messages
- Cluster graph for set of factors, each node *i* is associated with a subset (**cluster**) C_i.
  - Family-preserving: each factor's variables are completely embedded in a cluster

Clique tree properties

- **Sepset** $S_{ij} = C_i \cap C_j$
  - **separation** set: Variables **X** on one side of sepset are separated from the variables **Y** on the other side in the factor graph given variables in **S**
- **Running intersection property**
  - if C_i and C_j both contain X, then all cliques on the unique path between them also contain X

Clique trees

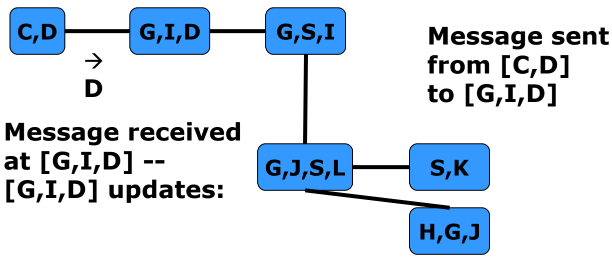


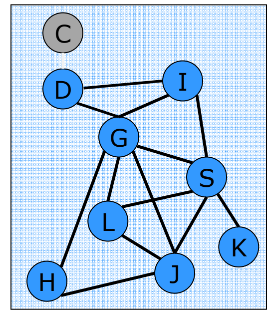
Running intersection:
E.g. Cliques involving S form a connected subtree.

Initial potentials π_i^0**:**
Assign factors to cliques and multiply them.

Message Passing VE

- Query for P(J)
  - Eliminate C: $\tau_1(D) = \sum_C \pi_1^0[C,D]$

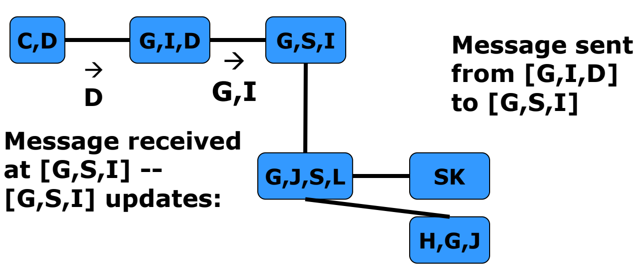


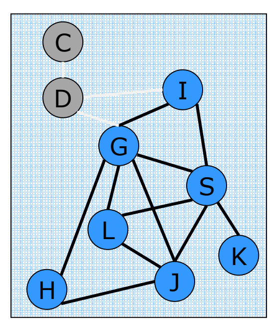


$$\pi_2[G,I,D] = \tau_1(D) \times \pi_2^0[G,I,D]$$

Message Passing VE

- Query for P(J)
  - Eliminate D: $\tau_2(G,I) = \sum_D \pi_2[G,I,D]$

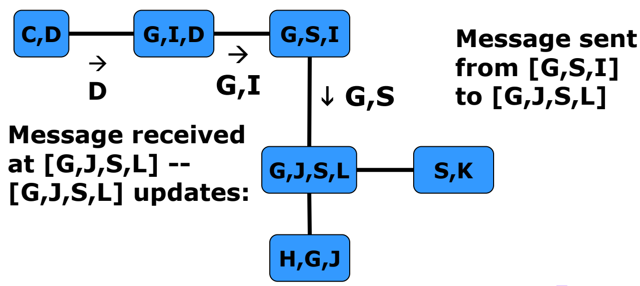


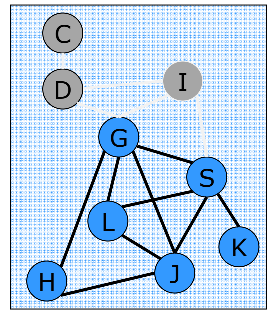


$$\pi_3[G,S,I] = \tau_2(G,I) \times \pi_3^0[G,S,I]$$

Message Passing VE

- Query for P(J)
  - Eliminate I: $\tau_3(G,S) = \sum_I \pi_3[G,S,I]$



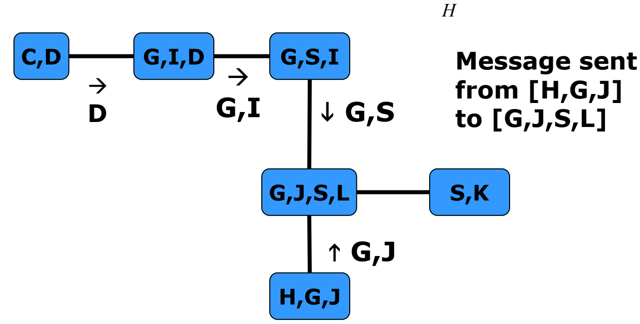


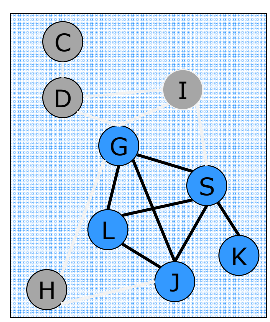
$$\pi_4[G,J,S,L] = \tau_3(G,S) \times \pi_4^0[G,J,S,L]$$!

[G,J,S,L] is not **ready**!

Message Passing VE

- Query for P(J)
  - Eliminate H: $\tau_4(G,J) = \sum_H \pi_5[H,G,J]$



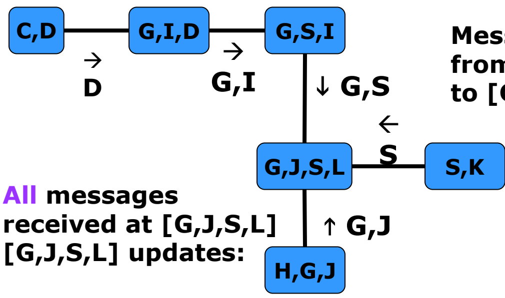


$$\pi_4[G,J,S,L] = \tau_3(G,S) \times \tau_4(G,J) \times \pi_4^0[G,J,S,L]$$

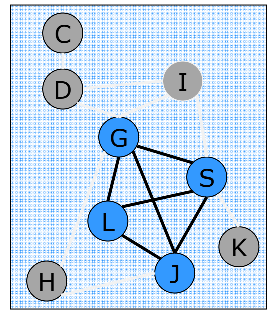
And …

Message Passing VE

- Query for P(J)
  - Eliminate K: $\tau_6(S) = \sum_K \pi^0[S,K]$



Message sent from [S,K] to [G,J,S,L]



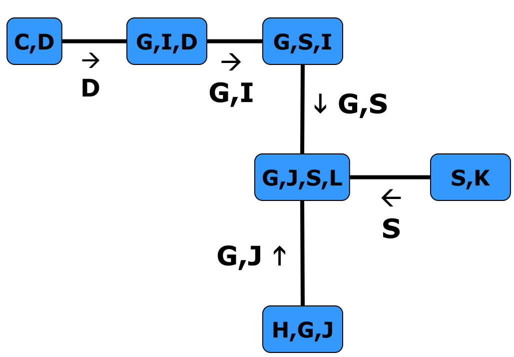
All messages received at [G,J,S,L]
[G,J,S,L] updates:

$$\pi_4[G,J,S,L] = \tau_3(G,S) \times \tau_4(G,J) \times \tau_6(S) \times \pi_4^0[G,J,S,L]$$

And calculate P(J) from it by summing out G,S,L

Message Passing VE

- [G,J,S,L] clique potential
- … is used to finish the inference

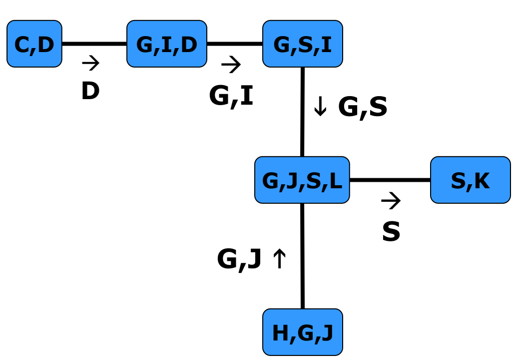


Message passing VE

- Often, **many marginals are desired**
  - Inefficient to re-run each inference from scratch
  - One distinct message per edge & direction
- **Methods :**
  - Compute (**unnormalized**) marginals for any vertex (clique) of the tree
  - Results in a ***calibrated* clique tree** $\sum_{C_i - S_{ij}} \pi_i = \sum_{C_j - S_{ij}} \pi_j$
- Recap: three kinds of factor objects
  - Initial potentials, final potentials and messages

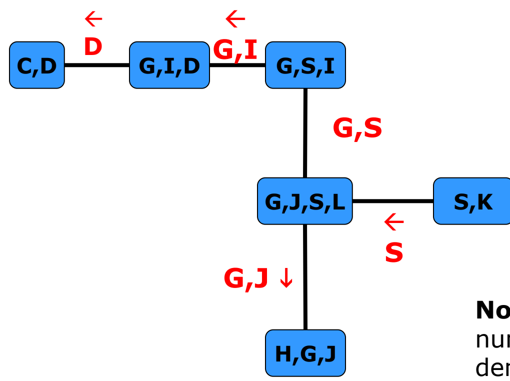
Two-pass message passing VE

- Chose the root clique, e.g. [S,K]
- Propagate messages to the root



Two-pass message passing VE

- Send messages back from the root



Notation:
number the cliques and denote the messages

$$\delta_{i \to j}$$

Message Passing: BP

- Graphical model of a **distribution**
  - More edges = larger expressive power
  - Clique tree also a model of distribution
  - Message passing preserves the model but changes the parameterization
- Different but equivalent algorithm

Factor division

A=1	B=1	0.5
A=1	B=2	0.4
A=2	B=1	0.8
A=2	B=2	0.2
A=3	B=1	0.6
A=3	B=2	0.5

A=1	0.4
A=2	0.4
A=3	0.5

A=1	B=1	0.5/0.4=1.25
A=1	B=2	0.4/0.4=1.0
A=2	B=1	0.8/0.4=2.0
A=2	B=2	0.2/0.4=2.0
A=3	B=1	0.6/0.5=1.2
A=3	B=2	0.5/0.5=1.0

Inverse of factor product

Message Passing: BP

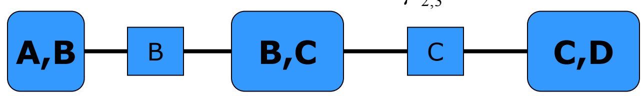
- Each node: multiply all the messages and divide by the one that is coming from node we are sending **the message to**
  - Clearly the same as VE

$$\delta_{i\to j} = \frac{\sum_{C_i - S_{ij}} \pi_i}{\delta_{j\to i}} = \frac{\sum_{C_i - S_{ij}} \prod_{k\in N(i)} \delta_{k\to i}}{\delta_{j\to i}} = \sum_{C_i - S_{ij}} \prod_{k\in N(i)\setminus j} \delta_{k\to i}$$

  - Initialize the messages on the edges to 1

Message Passing: BP

$\mu_{2,3} = 1$



$\pi_1^0(A,B)$ $\quad \pi_2^0(B,C)$ $\quad \longrightarrow \quad \pi_3^0(C,D)$

Store the last message on the edge and divide each passing message by the last stored.

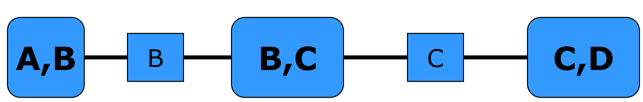
$$\delta_{2\to 3} = \left(\sum_B \pi_2^0(B,C)\right)$$

$$\pi_3(C,D) = \pi_3^0(C,D)\frac{\delta_{2->3}}{\mu_{2,3}} = \pi_3^0(C,D)\sum_B \pi_2^0(B,C)$$

$$\mu_{2,3} = \delta_{2\to 3} = \left(\sum_B \pi_2^0(B,C)\right)$$ New message

Message Passing: BP

$$\mu_{2,3} = \left(\sum_B \pi_2^0(B,C)\right)$$



$\pi_1^0(A,B)$ $\quad \pi_2^0(B,C)$ $\quad \longleftarrow \quad \pi_3(C,D)$

Store the last message on the edge and divide each passing message by the last stored.

$$\pi_3(C,D) = \pi_3^0(C,D)\sum_B \pi_2^0(B,C) = \pi_3^0(C,D)\mu_{2,3}$$

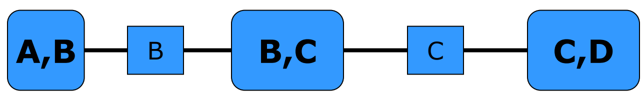
$$\delta_{3->2} = \left(\sum_D \pi_3(C,D)\right)$$

$$\pi_2(B,C) = \pi_2^0(B,C)\frac{\delta_{3->2}}{\mu_{2,3}(C)} = \frac{\pi_2^0(B,C)}{\mu_{2,3}(C)} \times \sum_D \pi_3^0(C,D) \times \mu_{2,3}(C) = \pi_2^0(B,C) \times \sum_D \pi_3^0(C,D)$$

$$\mu_{2,3} = \delta_{3->2} = \left(\sum_D \pi_3(C,D)\right) = \sum_D \pi_3^0(C,D)\sum_B \pi_2^0(B,C)$$ New message

Message Passing: BP

$$\mu_{2,3} = \sum_D \pi_3^0(C,D) \sum_B \pi_2^0(B,C)$$



$\pi_1^0(A,B)$ $\pi_2(B,C)$ ← $\pi_3(C,D)$

$$\pi_3(C,D) = \pi_3^0(C,D) \sum_B \pi_2^0(B,C)$$

Store the last message on the edge and divide each passing message by the last stored.

$$\delta_{3->2} = \left(\sum_D \pi_3(C,D) \right)$$

$$\pi_2(B,C) = \pi_2^0(B,C) \times \sum_D \pi_3^0(C,D)$$

The same as before

$$\pi_2(B,C) = \pi_2(B,C) \frac{\delta_{3->2}}{\mu_{2,3}(C)} = \pi_2(B,C) \times \frac{\sum_D \pi_3^0(C,D) \times \sum_B \pi_2^0(B,C)}{\sum_D \pi_3^0(C,D) \times \sum_B \pi_2^0(B,C)} = \pi_2(B,C)$$

Message Propagation: BP

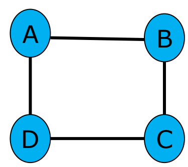
- **Lauritzen-Spiegelhalter algorithm**
- Two kinds of objects: clique and sepset potentials
  - Initial potentials not kept
- Improved "stability" of asynchronous algorithm (repeated messages cancel out)
- **New distribution representation**
  - clique tree potential

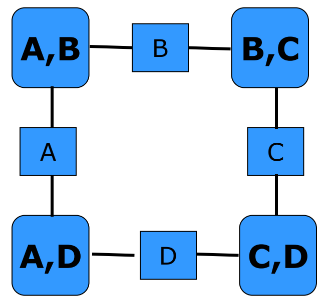
$$\pi_T = \frac{\prod_{C_i \in T} \pi_i(C_i)}{\prod_{(C_i \leftrightarrow C_j) \in T} \mu_{ij}(S_{ij})} = P_F(X)$$

  - Clique tree invariant = P_F

Loopy belief propagation

- The asynchronous BP algorithm works on clique trees
- What if we run the belief propagation algorithm on a non-tree structure?





- Sometimes converges
- If it converges it leads to an approximate solution
- **Advantage:** tractable for large graphs

Loopy belief propagation

- If the BP algorithm converges, it converges to an optimum of the Bethe free energy

See papers:

- Yedidia J.S., Freeman W.T. and Weiss Y. Generalized Belief Propagation, 2000
- Yedidia J.S., Freeman W.T. and Weiss Y. Understanding Belief Propagation and Its Generalizations, 2001