CS 2750 Machine Learning
Lecture 7

Density estimation III

Milos Hauskrecht
milos@pitt.edu
5329 Sennott Square

Exponential family

Exponential family:

- all probability mass / density functions that can be written in the exponential normal form

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x})\right]$$

- $\boldsymbol{\eta}$ a vector of natural (or canonical) parameters
- $t(\mathbf{x})$ a function referred to as a sufficient statistic
- $h(\mathbf{x})$ a function of x (it is less important)
- $Z(\boldsymbol{\eta})$ a normalization constant (a partition function)

$$Z(\boldsymbol{\eta}) = \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}$$

- Other common form:

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x}) - A(\boldsymbol{\eta})\right] \qquad \log Z(\boldsymbol{\eta}) = A(\boldsymbol{\eta})$$

Exponential family: examples

- **Bernoulli distribution**

$$p(x \mid \pi) = \pi^x (1-\pi)^{1-x}$$
$$= \exp\left\{ \log\left(\frac{\pi}{1-\pi} \right) x + \log(1-\pi) \right\}$$
$$= \exp\{\log(1-\pi)\} \exp\left\{ \log\left(\frac{\pi}{1-\pi} \right) x \right\}$$

- **Exponential family**

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x}) \right]$$

- **Parameters**

$\boldsymbol{\eta} = ?$ $\qquad$ $t(\mathbf{x}) = ?$

$Z(\boldsymbol{\eta}) = ?$ $\qquad$ $h(\mathbf{x}) = ?$

Exponential family: examples

- **Bernoulli distribution**

$$p(x \mid \pi) = \pi^x (1-\pi)^{1-x}$$
$$= \exp\left\{ \log\left(\frac{\pi}{1-\pi} \right) x + \log(1-\pi) \right\}$$
$$= \exp\{\log(1-\pi)\} \exp\left\{ \log\left(\frac{\pi}{1-\pi} \right) x \right\}$$

- **Exponential family**

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x}) \right]$$

- **Parameters**

$\boldsymbol{\eta} = \log \frac{\pi}{1-\pi}$ ← **logit function** $\qquad$ $t(\mathbf{x}) = x$

$Z(\boldsymbol{\eta}) = \frac{1}{1-\pi} = 1 + e^{\eta}$ $\qquad$ $h(\mathbf{x}) = 1$

Exponential family: examples

- **Univariate Gaussian distribution**

$$p(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mu}{2\sigma^2} - \log\sigma\right) \exp\left\{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2\right\}$$

- **Exponential family** $f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(x) \exp\left[\eta^T t(x)\right]$
- **Parameters**

$\boldsymbol{\eta} = ?$ $\quad t(\mathbf{x}) = ?$

$Z(\boldsymbol{\eta}) = ?$ $\quad h(\mathbf{x}) = ?$

Exponential family: examples

- **Univariate Gaussian distribution**

$$p(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mu}{2\sigma^2} - \log\sigma\right) \exp\left\{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2\right\}$$

- **Exponential family** $f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(x) \exp\left[\eta^T t(x)\right]$
- **Parameters**

$$\boldsymbol{\eta} = \begin{bmatrix} \mu/2\sigma^2 \\ -1/2\sigma^2 \end{bmatrix} \qquad t(\mathbf{x}) = \begin{bmatrix} x \\ x^2 \end{bmatrix}$$

$$Z(\boldsymbol{\eta}) = \exp\left\{\frac{\mu}{2\sigma^2} + \log\sigma\right\} = \exp\left\{-\frac{\eta_1^2}{4\eta_2} - \frac{1}{2}\log(-2\eta_2)\right\}$$

$$h(\mathbf{x}) = 1/\sqrt{2\pi}$$

Exponential family

- **For iid samples, the likelihood of data is**

$$P(D \mid \boldsymbol{\eta}) = \prod_{i=1}^{n} p(\mathbf{x}_i \mid \boldsymbol{\eta}) = \prod_{i=1}^{n} h(\mathbf{x}_i) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x}_i) - A(\boldsymbol{\eta})\right]$$

$$= \left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right] \exp\left[\sum_{i=1}^{n} \boldsymbol{\eta}^T t(\mathbf{x}_i) - A(\boldsymbol{\eta})\right]$$

$$= \left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right] \exp\left[\boldsymbol{\eta}^T \left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - nA(\boldsymbol{\eta})\right]$$

- **Important:**
  - the dimensionality of the sufficient statistic remains the same with the number of samples

Exponential family

- **The log likelihood of data is**

$$l(D, \boldsymbol{\eta}) = \log\left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right] \exp\left[\boldsymbol{\eta}^T \left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - nA(\boldsymbol{\eta})\right]$$

$$= \log\left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right] + \left[\boldsymbol{\eta}^T \left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - nA(\boldsymbol{\eta})\right]$$

- **Optimizing the loglikelihood**

$$\nabla_{\boldsymbol{\eta}} l(D, \boldsymbol{\eta}) = \left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - n\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) \quad = \mathbf{0}$$

- **For the ML estimate it must hold**

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \frac{1}{n}\left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right)$$

Exponential family

- **Rewritting the gradient:**

Exponential family

- **Rewritting the gradient:**

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \nabla_{\boldsymbol{\eta}} \log Z(\boldsymbol{\eta}) = \nabla_{\boldsymbol{\eta}} \log \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}$$

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \frac{\int t(\mathbf{x}) h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}}{\int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}}$$

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \int t(\mathbf{x}) h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x}) - A(\boldsymbol{\eta})\right\} d\mathbf{x}$$

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = E(t(\mathbf{x}))$$

- **Result:** $$E(t(\mathbf{x})) = \frac{1}{n}\left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right)$$
- **For the ML estimate the parameters $\boldsymbol{\eta}$ should be adjusted such that the expectation of the statistic t(x) is equal to the observed sample statistics**

Moments of the distribution

- **For the exponential family**
  - The k-th moment of the statistic corresponds to the k-th derivative of $A(\boldsymbol{\eta})$
  - If x is a component of t(x) then we get the moments of the distribution by differentiating its corresponding natural parameter
- **Example: Bernoulli** $p(x \mid \pi) = \exp\left\{\log\left(\frac{\pi}{1-\pi}\right)x + \log(1-\pi)\right\}$

  $A(\boldsymbol{\eta}) = \log\frac{1}{1-\pi} = \log(1+e^{\eta})$
- **Derivatives:**

$$\frac{\partial A(\boldsymbol{\eta})}{\partial \eta} = \frac{\partial}{\partial \eta}\log(1+e^{\eta}) = \frac{e^{\eta}}{(1+e^{\eta})} = \frac{1}{(1+e^{-\eta})} = \pi$$

$$\frac{\partial A(\boldsymbol{\eta})}{\partial \eta^2} = \frac{\partial}{\partial \eta}\frac{1}{(1+e^{-\eta})} = \pi(1-\pi)$$

Non-parametric density estimation

Nonparametric Density Estimation

- **Parametric distribution models** are:
  - restricted to specific functional forms, which may not always be suitable;
  - **Example:** modeling a multimodal distribution with a single, unimodal model.

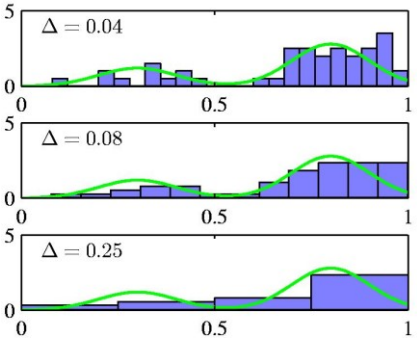
- **Nonparametric approaches:**
  - Do not make any strong assumption about the overall shape of the distribution being modelled.

Nonparametric Methods

Histogram methods:

partition the data space into distinct bins with widths Δ_i and count the number of observations, n_i, in each bin.

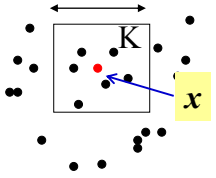
$$p_i = \frac{n_i}{N\Delta_i}$$



- Often, the same width is used for all bins, $\Delta_i = \Delta$.
- Δ acts as a smoothing parameter.
- Binning does not work well in the in a d-dimensional space,

Nonparametric Methods

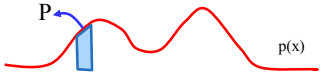
- Binning does not work well in the in a d-dimensional space,
  - M bins in each dimension will require M^d bins!
- **Solution:**
  - Build the estimates of p(**x**) by considering the data points in D and how similar (or close) they are to **x**
  - **Example: Parzen window**
    - As if we build a bin dynamically for **x** for which we need p(**x**)



Nonparametric Methods

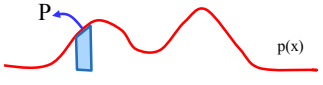
- Assume observations drawn from a density p(x) and consider a small region R containing x such that

$$P = \int_R p(x)dx$$



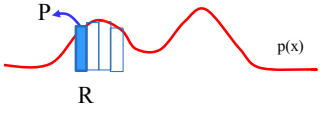
- The probability that K out of N observations lie inside R is *Bin(K,N,P)* and if N is large

$$K \cong NP$$



If the volume of R, V, is sufficiently small, p(x) is approximately constant over R and

$$P \cong p(x)V$$



Thus

$$p(x) = \frac{P}{V}$$

Putting things together we get:

$$p(x) = \frac{K}{NV}$$

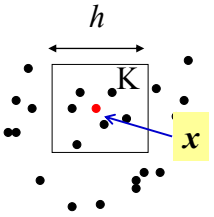
Nonparametric methods: kernel methods

Solution 1: Estimate the probability for **x** based on the fixed volume **V** built around **x**

$$p(x) = \frac{K}{NV}$$

- **Fix V, estimate K from the data**

Example: Parzen window



Nonparametric methods: kernel methods

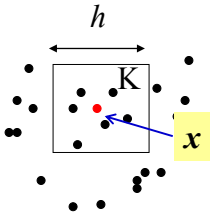
Kernel Density Estimation:

- **Parzen window:** Let R be a hypercube centred on **x** that defines the **kernel function:**

$$k\left(\frac{x - x_n}{h}\right) = \begin{cases} 1 & |(x_i - x_{ni})|/h \le 1/2 \\ 0 & otherwise \end{cases} \qquad i = 1, \ldots D$$

- **It follows that**

$$K = \sum_{n=1}^{N} k\left(\frac{x - x_n}{h}\right)$$



- **and hence**

$$p(x) = \frac{K}{NV} = \frac{1}{Nh^D} \sum_{n=1}^{N} k\left(\frac{x - x_n}{h}\right)$$

Nonparametric Methods: smooth kernels

To avoid discontinuities in p(x) because of sharp boundaries we can use a **smooth kernel**, e.g. a Gaussian

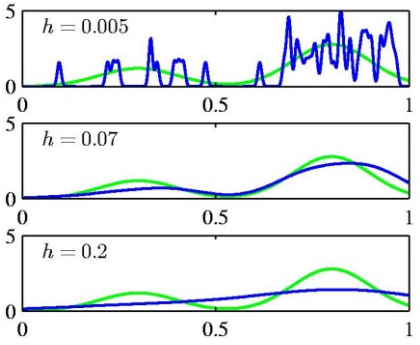
$$p(\mathbf{x}) = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{(2\pi h^2)^{D/2}}\exp\left[-\frac{\|\mathbf{x}-\mathbf{x}_n\|^2}{2h^2}\right]$$

- Any kernel such that

$$k(\mathbf{u}) \geq 0$$

$$\int k(\mathbf{u})d\mathbf{u} = 1$$

- will work.



h acts as a smoother.

Nonparametric Methods: kNN estimation

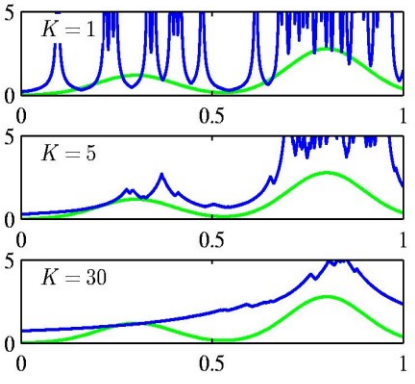
Solution 2: Estimate the probability for **x** based on a fixed count **K** for a variable volume **V** built around **x**

fix K, estimate V from the data

Nearest Neighbour Density Estimation:

Consider a hyper-sphere centred on x and let it grow to a volume, V*, that includes K of the given N data points. Then

$$p(\mathbf{x}) \simeq \frac{K}{NV^\star}.$$



K acts as a smoother

Nonparametric vs Parametric Methods

Nonparametric models:

- More flexibility – no density model is needed
- But require storing the entire dataset
- and the computation is performed with all data examples.

Parametric models:

- Once fitted, only parameters need to be stored
- They are much more efficient in terms of computation
- But the model needs to be picked in advance