CS 2750 Machine Learning
Lecture 7

Classification learning:
• Logistic regression
• Generative classification model

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Classification

- **Data:** $D = \{d_1, d_2, .., d_n\}$
  $d_i = < \mathbf{x}_i, y_i >$
  - y_i **represents a discrete class value**
- **Goal: learn** $f : X \rightarrow Y$
- **Binary classification**
  - **A special case when** $Y \in \{0,1\}$
- **First step:**
  - we need to devise a model of the function f

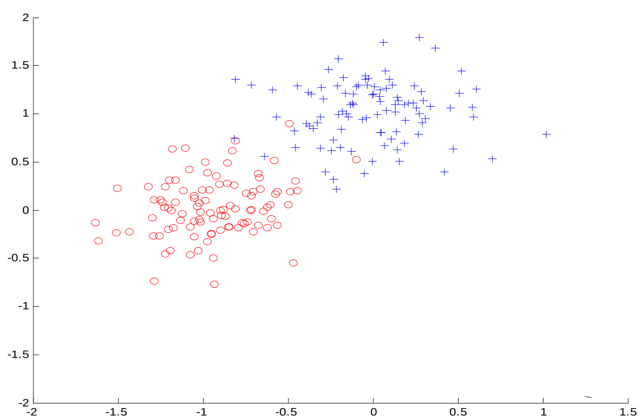
Discriminant functions

- A common way to represent a **classifier is by using**
  - **Discriminant functions**
- **Works for both the binary and multi-way classification**
- **Idea:**
  - For every class $i = 0,1, \ldots k$ define a function $g_i(\mathbf{x})$ mapping $X \rightarrow \Re$
  - When the decision on input **x** should be made choose the class with the highest value of $g_i(\mathbf{x})$

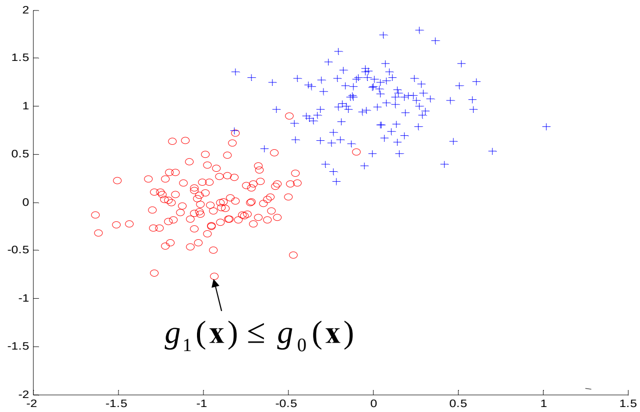
$$y^* = \arg\max_i \; g_i(\mathbf{x})$$

- So what happens with the input space? Assume a binary case.

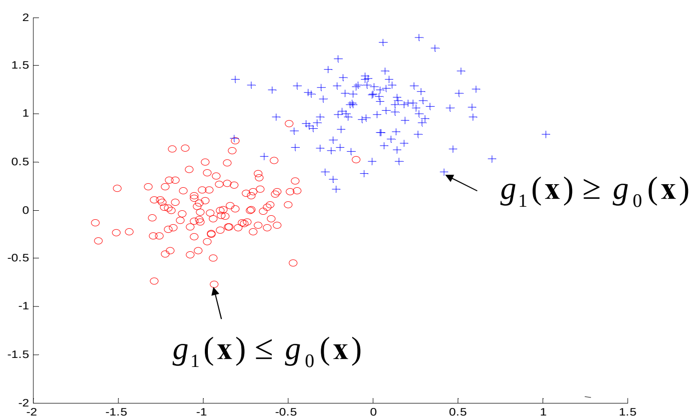
Discriminant functions



Discriminant functions

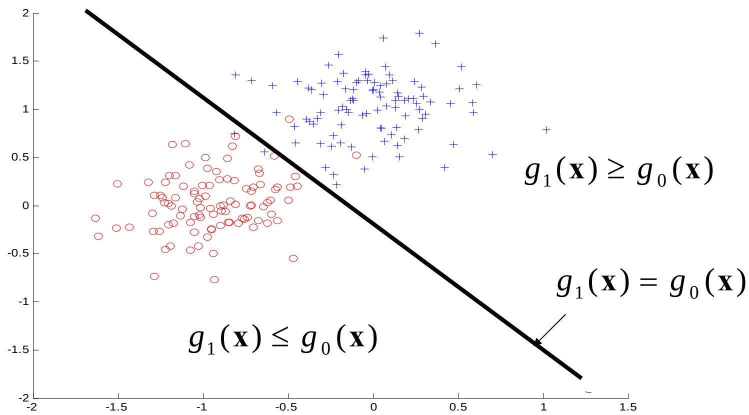


Discriminant functions

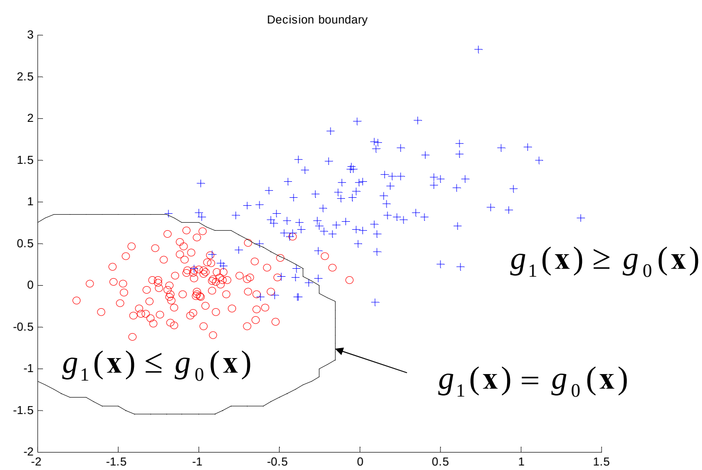


Discriminant functions

- **Decision boundary:** discriminant functions are equal



Quadratic decision boundary



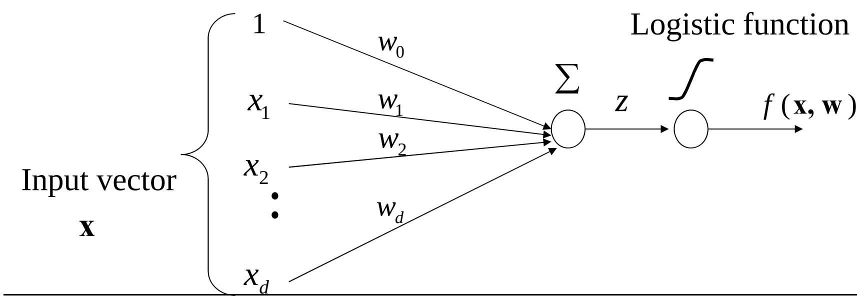
Logistic regression model

- **Defines a linear decision boundary**
- **Discriminant functions:**

$$g_1(\mathbf{x}) = g(\mathbf{w}^T\mathbf{x}) \qquad\qquad g_0(\mathbf{x}) = 1 - g(\mathbf{w}^T\mathbf{x})$$

- **where** $g(z) = 1/(1+e^{-z})$ - is a logistic function

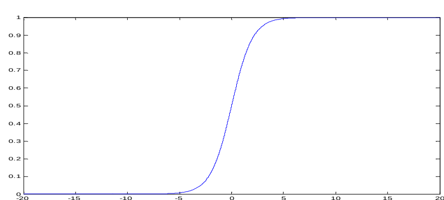
$$f(\mathbf{x},\mathbf{w}) = g_1(\mathbf{w}^T\mathbf{x}) = g(\mathbf{w}^T\mathbf{x})$$



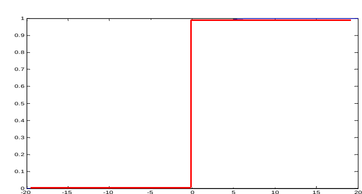
Logistic function

Function: $$g(z) = \frac{1}{(1+e^{-z})}$$

- Is also referred to as a **sigmoid function**
- takes a real number and outputs the number in the interval [0,1]
- Models a smooth switching function; replaces hard threshold function



Logistic (smooth) switching



Threshold (hard) switching

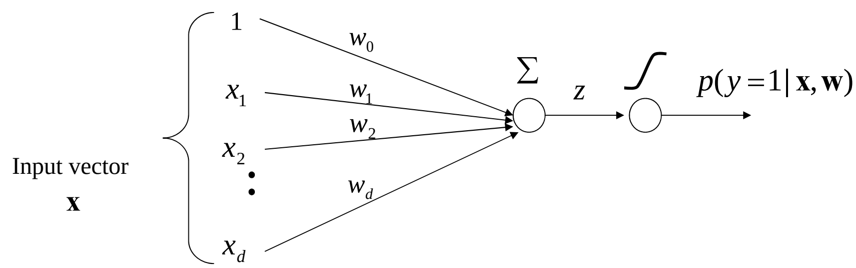
Logistic regression model

- **Discriminant functions:**

  $$g_1(\mathbf{x}) = g(\mathbf{w}^T\mathbf{x}) \qquad g_0(\mathbf{x}) = 1 - g(\mathbf{w}^T\mathbf{x})$$

- **Values of discriminant functions vary in interval [0,1]**
  - **Probabilistic interpretation**

  $$f(\mathbf{x},\mathbf{w}) = p(y=1 \mid \mathbf{w},\mathbf{x}) = g_1(\mathbf{x}) = g(\mathbf{w}^T\mathbf{x})$$



Logistic regression

- We learn **a probabilistic function**

  $$f : X \rightarrow [0,1]$$

  - where f describes the probability of class 1 given $\mathbf{x}$

  $$f(\mathbf{x},\mathbf{w}) = g_1(\mathbf{w}^T\mathbf{x}) = p(y=1 \mid \mathbf{x},\mathbf{w})$$

Note that: $$p(y=0 \mid \mathbf{x},\mathbf{w}) = 1 - p(y=1 \mid \mathbf{x},\mathbf{w})$$

- Making decisions with the logistic regression model:

  ?

Logistic regression

- We learn **a probabilistic function**

$$f : X \to [0,1]$$

  – where f describes the probability of class 1 given $\mathbf{x}$

$$f(\mathbf{x}, \mathbf{w}) = g_1(\mathbf{w}^T \mathbf{x}) = p(y=1 \mid \mathbf{x}, \mathbf{w})$$

Note that:

$$p(y=0 \mid \mathbf{x}, \mathbf{w}) = 1 - p(y=1 \mid \mathbf{x}, \mathbf{w})$$

- Making decisions with the logistic regression model:

$$\boxed{\begin{array}{l} \text{If } \; p(y=1 \mid \mathbf{x}) \geq 1/2 \quad \text{then choose } \mathbf{1} \\ \text{Else choose } \mathbf{0} \end{array}}$$

Linear decision boundary

- Logistic regression model defines a **linear decision boundary**
- **Why?**
- **Answer:** Compare two **discriminant functions**.
- **Decision boundary:** $g_1(\mathbf{x}) = g_0(\mathbf{x})$
- For the boundary it must hold:

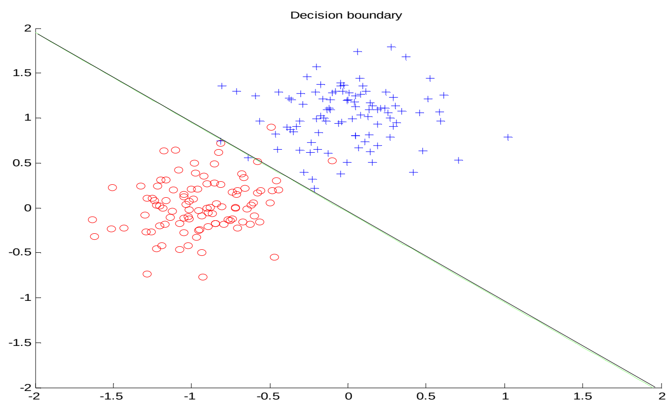
$$\log \frac{g_o(\mathbf{x})}{g_1(\mathbf{x})} = \log \frac{1 - g(\mathbf{w}^{\mathbf{T}} \mathbf{x})}{g(\mathbf{w}^{\mathbf{T}} \mathbf{x})} = 0$$

$$\log \frac{g_o(\mathbf{x})}{g_1(\mathbf{x})} = \log \frac{\dfrac{\exp -(\mathbf{w}^{\mathbf{T}} \mathbf{x})}{1 + \exp -(\mathbf{w}^{\mathbf{T}} \mathbf{x})}}{\dfrac{1}{1 + \exp -(\mathbf{w}^{\mathbf{T}} \mathbf{x})}} = \log \exp -(\mathbf{w}^{\mathbf{T}} \mathbf{x}) = \mathbf{w}^{\mathbf{T}} \mathbf{x} = 0$$

Logistic regression model. Decision boundary

- **LR defines a linear decision boundary**

  Example: 2 classes (blue and red points)



Logistic regression: parameter learning

Likelihood of outputs

- **Let**

$$D_i = < \mathbf{x}_i, y_i > \qquad \mu_i = p(y_i = 1 | \mathbf{x}_i, \mathbf{w}) = g(z_i) = g(\mathbf{w}^T \mathbf{x})$$

- **Then**

$$L(D, \mathbf{w}) = \prod_{i=1}^{n} P(y = y_i | \mathbf{x}_i, \mathbf{w}) = \prod_{i=1}^{n} \mu_i^{y_i} (1 - \mu_i)^{1-y_i}$$

- **Find weights w that maximize the likelihood of outputs**
  - Apply the log-likelihood trick. The optimal weights are the same for both the likelihood and the log-likelihood

$$l(D, \mathbf{w}) = \log \prod_{i=1}^{n} \mu_i^{y_i} (1 - \mu_i)^{1-y_i} = \sum_{i=1}^{n} \log \mu_i^{y_i} (1 - \mu_i)^{1-y_i} =$$

$$= \sum_{i=1}^{n} y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)$$

Logistic regression: parameter learning

- **Notation:** $\mu_i = p(y_i = 1 | \mathbf{x}_i, \mathbf{w}) = g(z_i) = g(\mathbf{w}^T \mathbf{x})$
- **Log likelihood**

$$l(D, \mathbf{w}) = \sum_{i=1}^{n} y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)$$

- **Derivatives of the loglikelihood**

$$-\frac{\partial}{\partial w_j} l(D, \mathbf{w}) = \sum_{i=1}^{n} - x_{i,j} (y_i - g(z_i))$$

Nonlinear in weights !!

$$\nabla_{\mathbf{w}} - l(D, \mathbf{w}) = \sum_{i=1}^{n} -\mathbf{x}_i (y_i - g(\mathbf{w}^T \mathbf{x}_i)) = \sum_{i=1}^{n} -\mathbf{x}_i (y_i - f(\mathbf{w}, \mathbf{x}_i))$$

- **Gradient descent:**

$$\mathbf{w}^{(k)} \leftarrow \mathbf{w}^{(k-1)} - \alpha(k) \nabla_{\mathbf{w}} [-l(D, \mathbf{w})] |_{\mathbf{w}^{(k-1)}}$$

$$\mathbf{w}^{(k)} \leftarrow \mathbf{w}^{(k-1)} + \alpha(k) \sum_{i=1}^{n} [y_i - f(\mathbf{w}^{(k-1)}, \mathbf{x}_i)] \mathbf{x}_i$$

Derivation of the gradient

- **Log likelihood** $l(D, \mathbf{w}) = \sum_{i=1}^{n} y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)$

- **Derivatives of the loglikelihood**

$$\frac{\partial}{\partial w_j} l(D, \mathbf{w}) = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} [y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)] \frac{\partial z_i}{\partial w_j}$$

$$\frac{\partial z_i}{\partial w_j} = x_{i,j}$$

Derivative of a logistic function

$$\frac{\partial g(z_i)}{\partial z_i} = g(z_i)(1 - g(z_i))$$

$$\frac{\partial}{\partial z_i} [y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)] = y_i \frac{1}{g(z_i)} \frac{\partial g(z_i)}{\partial z_i} + (1 - y_i) \frac{-1}{1 - g(z_i)} \frac{\partial g(z_i)}{\partial z_i}$$

$$= y_i (1 - g(z_i)) + (1 - y_i)(-g(z_i)) = \boxed{y_i - g(z_i)}$$

$$\boxed{\nabla_{\mathbf{w}} l(D, \mathbf{w}) = \sum_{i=1}^{n} \mathbf{x}_i (y_i - g(\mathbf{w}^T \mathbf{x}_i)) = \sum_{i=1}^{n} \mathbf{x}_i (y_i - f(\mathbf{w}, \mathbf{x}_i))}$$

Logistic regression. Online gradient descent

- **On-line component of the loglikelihood**

$$-J_{\text{online}}(D_i, \mathbf{w}) = y_i \log \mu_i + (1 - y_i) \log(1 - \mu_i)$$

- **On-line learning update for weight w** $J_{online}(D_k, \mathbf{w})$

$$\mathbf{w}^{(k)} \leftarrow \mathbf{w}^{(k-1)} - \alpha(k) \nabla_{\mathbf{w}} [J_{online}(D_k, \mathbf{w})] |_{\mathbf{w}^{(k-1)}}$$

- **ith update for the logistic regression** and $D_k =< \mathbf{x}_k, y_k >$

$$\mathbf{w}^{(i)} \leftarrow \mathbf{w}^{(k-1)} + \alpha(k)[y_i - f(\mathbf{w}^{(k-1)}, \mathbf{x}_k)]\mathbf{x}_k$$

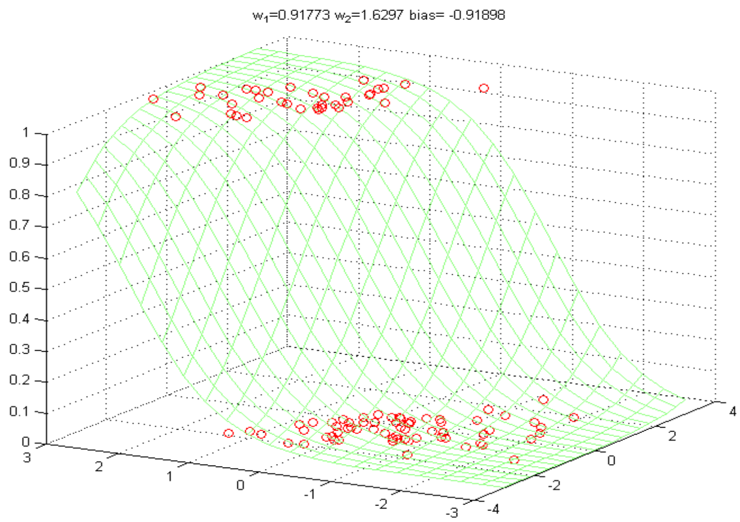
Online logistic regression algorithm

Online-logistic-regression (*D, number of iterations*)
    initialize weights $\mathbf{w} = (w_0, w_1, w_2 \ldots w_d)$
    for *i*=1:1*: number of iterations*
        do **select** a data point $D_i =< \mathbf{x}_i, y_i >$ from *D*
            set $\alpha = 1/i$
            update weights (in parallel)

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha(i)[y_i - f(\mathbf{w}, \mathbf{x}_i)]\mathbf{x}_i$$

    end for
    return weights $\mathbf{w}$

Online algorithm. Example.



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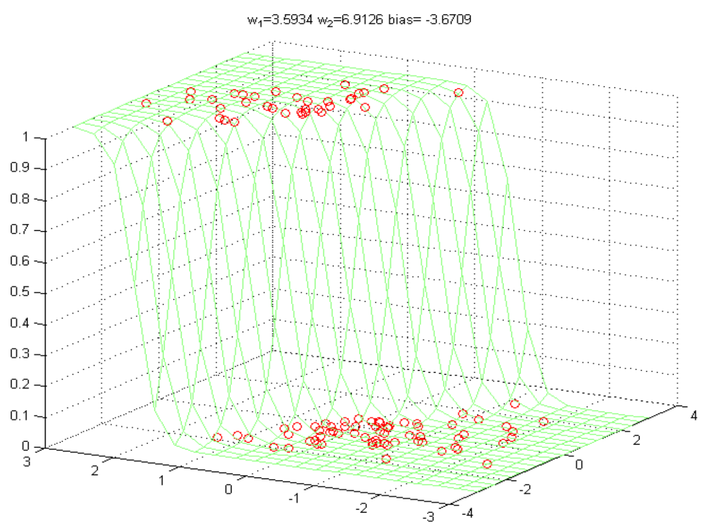
Online algorithm. Example.



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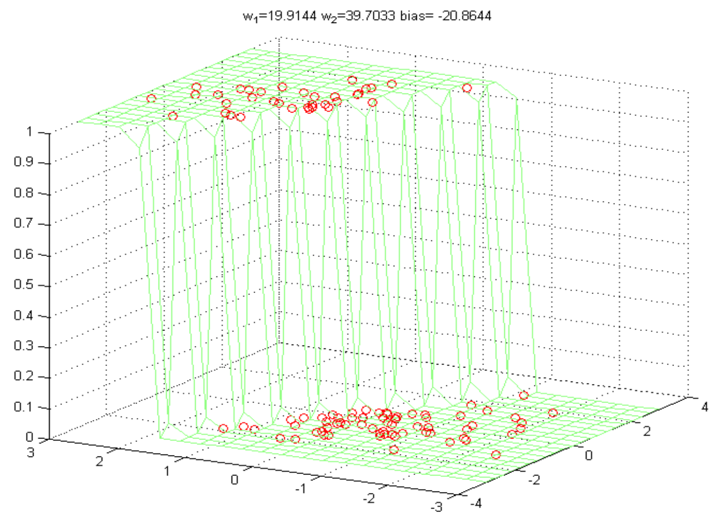
Online algorithm. Example.



Generative approach to classification

Logistic regression: model $p(y \mid \mathbf{x})$

Generative approach:

1. **Represent and learn the distribution** $p(\mathbf{x}, y)$
2. **Use it to define probabilistic discriminant functions**

   E.g. $g_o(\mathbf{x}) = p(y = 0 \mid \mathbf{x})$ $\quad g_1(\mathbf{x}) = p(y = 1 \mid \mathbf{x})$

Typical model $p(\mathbf{x}, y) = p(\mathbf{x} \mid y)\, p(y)$

- $p(\mathbf{x} \mid y)$ = **Class-conditional distributions (densities)**

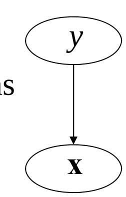
  binary classification: two class-conditional distributions

  $p(\mathbf{x} \mid y = 0)$ $\quad p(\mathbf{x} \mid y = 1)$

- $p(y)$ = **Priors on classes** - probability of class y

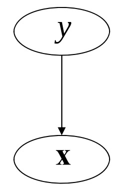
  binary classification: Bernoulli distribution

$$p(y = 0) + p(y = 1) = 1$$



Quadratic discriminant analysis (QDA)

Model:



- **Class-conditional distributions**
  - **multivariate normal distributions**

$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \quad \text{for} \quad y = 0$$

$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \quad \text{for} \quad y = 1$$

Multivariate normal $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

- **Priors on classes (class 0,1)** $y \sim Bernoulli$
  - **Bernoulli distribution**

$$p(y, \theta) = \theta^y (1 - \theta)^{1-y} \qquad y \in \{0,1\}$$

Learning of parameters of the QDA model

Density estimation in statistics

- We see examples – we do not know the parameters of Gaussians (class-conditional densities)

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

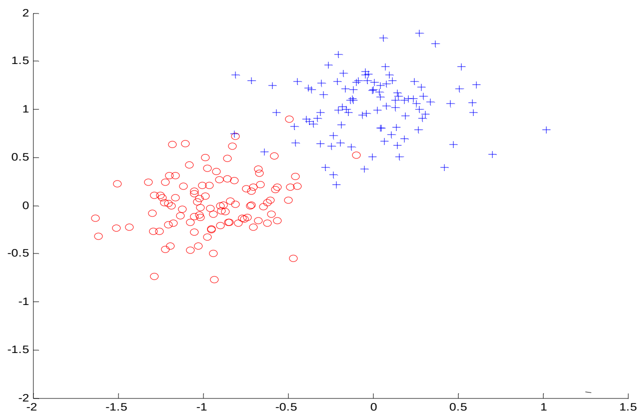
- **ML estimate of parameters** of a multivariate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for a set of n examples of $\mathbf{x}$

  Optimize log-likelihood: $l(D, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \log \prod_{i=1}^{n} p(\mathbf{x}_i \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$

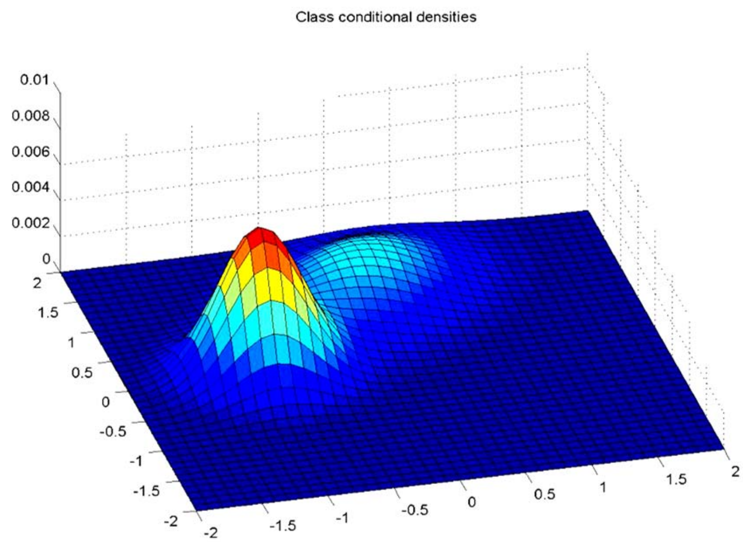
$$\hat{\boldsymbol{\mu}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T$$

- How about **class priors**?

QDA



2 Gaussian class-conditional densities



QDA: Making class decision

Basically we need to design discriminant functions

Two possible choices:

- **Likelihood of data** – choose the class (Gaussian) that explains the input data ($\mathbf{x}$) better (likelihood of the data)

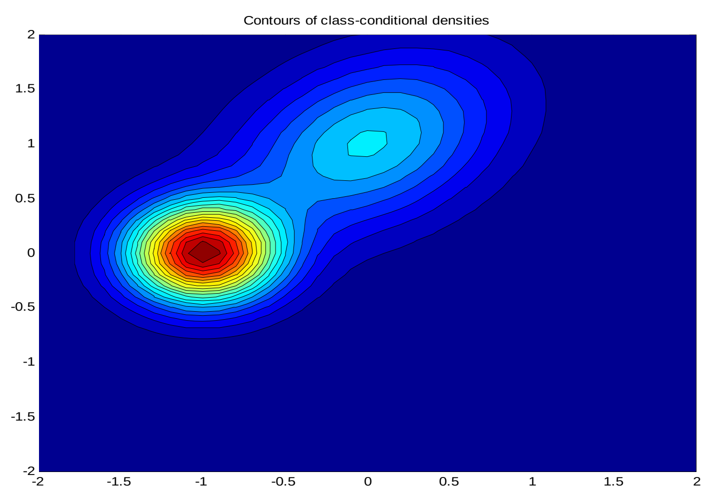
$$\underbrace{p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1)}_{g_1(\mathbf{x})} > \underbrace{p(\mathbf{x} \mid \mu_0, \mathbf{\Sigma}_0)}_{g_0(\mathbf{x})} \Rightarrow \begin{array}{l} \text{then } \ y=1 \\ \text{else } \ y=0 \end{array}$$

- **Posterior of a class** – choose the class with better posterior probability

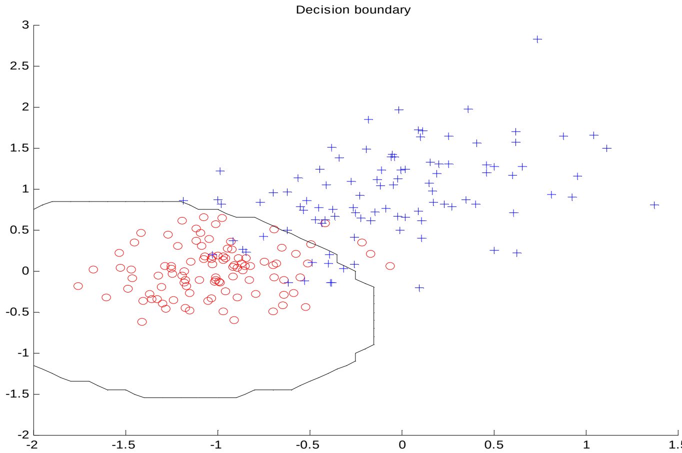
$$p(y=1 \mid \mathbf{x}) > p(y=0 \mid \mathbf{x}) \qquad \begin{array}{l} \text{then } \ y=1 \\ \text{else } \ y=0 \end{array}$$

$$p(y=1 \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1)\, p(y=1)}{p(\mathbf{x} \mid \mu_0, \mathbf{\Sigma}_0)\, p(y=0) + p(\mathbf{x} \mid \mu_1, \mathbf{\Sigma}_1)\, p(y=1)}$$

QDA: Quadratic decision boundary



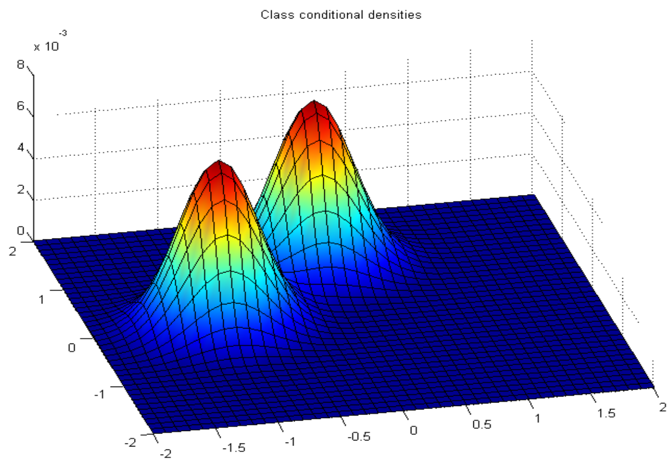
QDA: Quadratic decision boundary



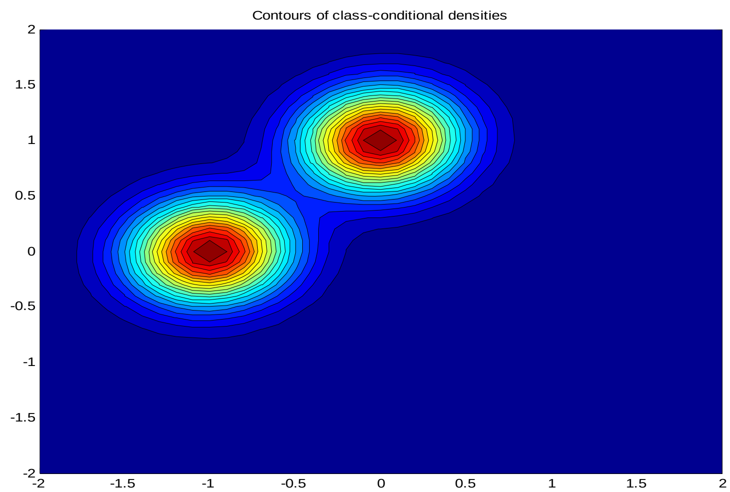
Linear discriminant analysis (LDA)

- When covariances are the same

$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}),\ y = 0$$
$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}),\ y = 1$$



LDA: Linear decision boundary



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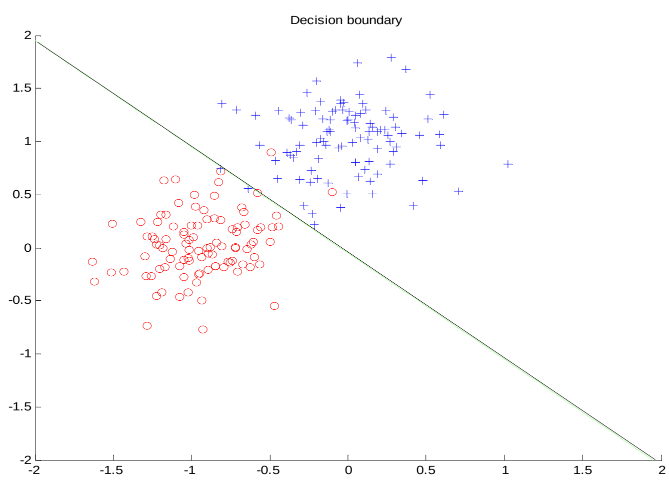
LDA: linear decision boundary



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