CS 2750 Machine Learning
Lecture 4

Density estimation

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Parametric density estimation

Parametric density estimation:

- A set of random variables $\mathbf{X} = \{X_1, X_2, \ldots, X_d\}$
- **A model of the distribution** over variables in $\boldsymbol{X}$ with **parameters** Θ : $\hat{p}(\mathbf{X} | \Theta)$
- **Data** $D = \{D_1, D_2, .., D_n\}$

Objective: find parameters Θ such that $p(\mathbf{X} | \Theta)$ describes data D the best

Parameter estimation (learning)

- **Maximum likelihood (ML)**

$$\Theta_{ML} = \arg\max_{\Theta} p(D \mid \Theta, \xi)$$

- **Maximum a posteriori probability (MAP)**

$$\Theta_{MAP} = \arg\max_{\Theta} p(\Theta \mid D, \xi)$$

- **Bayesian parameter estimation**
  - **use the posterior density**

$$p(\Theta \mid D, \xi)$$

- **Expected value**

$$\Theta_{EXP} = \int_{\Theta} \Theta p(\Theta \mid D, \xi) d\Theta$$

Probability of a binary outcome

Data: D a sequence of outcomes x_i such that

- **head** $x_i = 1$
- **tail** $x_i = 0$

Model: probability of a head θ
probability of a tail $(1-\theta)$

Assume: we know the probability θ

Probability of an outcome of a coin flip x_i

$$P(x_i \mid \theta) = \theta^{x_i}(1-\theta)^{(1-x_i)}$$ ← **Bernoulli distribution**

- Combines the probability of a head and a tail
- So that x_i is going to pick its correct probability
- Gives θ for $x_i = 1$
- Gives $(1-\theta)$ for $x_i = 0$

The goodness of fit to the data

Learning: **we do not know the value of the parameter** θ

Our learning goal:

- Find the parameter θ that fits the data D the best?

One solution to the "best": Maximize the likelihood

$$\Theta_{ML} = \arg\max_{\Theta} p(D \mid \Theta, \xi)$$

Intuition:

- more likely are the data given the model, the better is the fit

Assuming a sequence of n Bernoulli trials:

$$P(D \mid \theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{(1-x_i)}$$

Maximum likelihood (ML) estimate.

Likelihood of data:

$$P(D \mid \theta, \xi) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{(1-x_i)}$$

Maximum likelihood estimate

$$\theta_{ML} = \arg\max_{\theta} P(D \mid \theta, \xi)$$

Optimize log-likelihood (the same as maximizing likelihood)

$$\theta_{ML} = \arg\max_{\theta} \log P(D \mid \theta, \xi)$$

ML Solution: $\theta_{ML} = \frac{N_1}{N} = \frac{N_1}{N_1 + N_2}$

N_1 - number o heads seen $\quad N_2$ - number of tails seen

Maximum a posteriori estimate

Maximum a posteriori estimate

– Selects the mode of the **posterior distribution**

$$\theta_{MAP} = \arg\max_{\theta} p(\theta \mid D, \xi)$$

Likelihood of data **prior**

$$p(\theta \mid D, \xi) = \frac{P(D \mid \theta, \xi)\, p(\theta \mid \xi)}{P(D \mid \xi)}$$ **(via Bayes rule)**

Normalizing factor

$$P(D \mid \theta, \xi) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{(1-x_i)} = \theta^{N_1} (1-\theta)^{N_2}$$

$p(\theta \mid \xi)$ - is the prior probability on θ

How to choose the prior probability?

Prior distribution

Choice of prior: Beta distribution

$$p(\theta \mid \xi) = Beta(\theta \mid \alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \theta^{\alpha_1 - 1} (1-\theta)^{\alpha_2 - 1}$$

$\Gamma(x)$ - a Gamma function $\Gamma(x) = (x-1)\Gamma(x-1)$

For integer values of x $\Gamma(n) = (n-1)!$

Why to use Beta distribution?

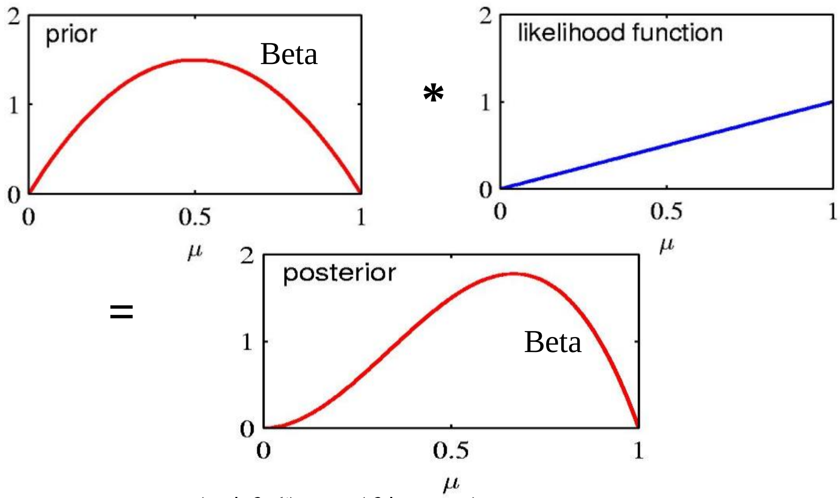
Beta distribution "**fits**" Bernoulli trials - **conjugate choices**

$$P(D \mid \theta, \xi) = \theta^{N_1} (1-\theta)^{N_2}$$

Posterior distribution is again a Beta distribution

$$p(\theta \mid D, \xi) = \frac{P(D \mid \theta, \xi) Beta(\theta \mid \alpha_1, \alpha_2)}{P(D \mid \xi)} = Beta(\theta \mid \alpha_1 + N_1, \alpha_2 + N_2)$$

Posterior distribution



$$p(\theta \mid D,\xi) = \frac{P(D \mid \theta,\xi) Beta(\theta \mid \alpha_1,\alpha_2)}{P(D \mid \xi)} = Beta(\theta \mid \alpha_1 + N_1, \alpha_2 + N_2)$$

Maximum a posterior probability

Maximum a posteriori estimate

– Selects the mode of the **posterior distribution**

$$\Theta_{MAP} = \arg\max_{\Theta} p(\Theta \mid D,\xi)$$

$$p(\theta \mid D,\xi) = \frac{P(D \mid \theta,\xi) Beta(\theta \mid \alpha_1,\alpha_2)}{P(D \mid \xi)} = Beta(\theta \mid \alpha_1 + N_1, \alpha_2 + N_2)$$

$$= \frac{\Gamma(\alpha_1 + \alpha_2 + N_1 + N_2)}{\Gamma(\alpha_1 + N_1)\Gamma(\alpha_2 + N_2)} \theta^{N_1 + \alpha_1 - 1}(1-\theta)^{N_2 + \alpha_2 - 1}$$

Notice that parameters of the prior
act like counts of heads and tails
(sometimes they are also referred to as **prior counts**)

MAP Solution: $\theta_{MAP} = \dfrac{\alpha_1 + N_1 - 1}{\alpha_1 + \alpha_2 + N_1 + N_2 - 2}$

Bayesian framework

Both ML or MAP estimates pick one value of the parameter

- **Assume:** there are two different parameter settings that are close in terms of their probability values. Using only one of them may introduce a strong bias, if we use them, for example, for predictions.

Bayesian parameter estimate

  - Remedies the limitation of one choice
  - Keeps all possible parameter values
  - Where $p(\theta \mid D, \xi) \approx Beta(\theta \mid \alpha_1 + N_1, \alpha_2 + N_2)$

- **The posterior can be used to define** $p(A \mid D)$**:**

$$p(A \mid D) = \int_{\mathbf{\Theta}} p(A \mid \mathbf{\Theta}) p(\mathbf{\Theta} \mid D, \xi) d\mathbf{\Theta}$$

Binomial distribution

Example problem: a biased coin

Outcomes: two possible values -- head or tail

Data: a set of order-independent outcomes for N trials

N_1 - number of heads seen N_2 - number of tails seen

can be calculated from the trial data !!!

Model: probability of a head θ

probability of a tail $(1-\theta)$

Probability of an outcome

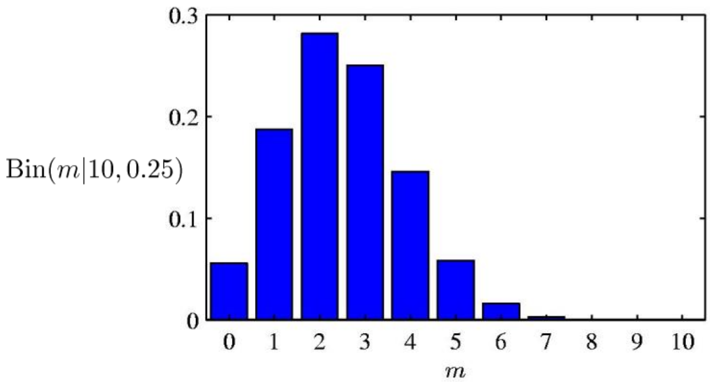
$$P(N_1 \mid N, \theta) = \binom{N}{N_1} \theta^{N_1} (1-\theta)^{N-N_1}$$ **Binomial distribution**

Objective:

We would like to estimate the probability of a **head** $\hat{\theta}$

Binomial distribution

Binomial distribution:



Maximum likelihood (ML) estimate.

Likelihood of data:

$$P(D \mid \theta) = \binom{N}{N_1} \theta^{N_1} (1-\theta)^{N_2} = \frac{N!}{N_1! N_2!} \theta^{N_1} (1-\theta)^{N_2}$$

Log-likelihood

$$l(D,\theta) = \log \binom{N}{N_1} \theta^{N_1} (1-\theta)^{N_2} = \underline{\log \frac{N!}{N_1! N_2!}} + N_1 \log \theta + N_2 \log(1-\theta)$$

Constant from the point of optimization !!!

ML Solution: $\theta_{ML} = \frac{N_1}{N} = \frac{N_1}{N_1 + N_2}$

The same as for Bernoulli and D with iid sequence of examples

Posterior density

Posterior density

$$p(\theta \mid D,\xi) = \frac{P(D \mid \theta,\xi)\,p(\theta \mid \xi)}{P(D \mid \xi)}$$ **(via Bayes rule)**

Prior choice

$$p(\theta \mid \xi) = Beta(\theta \mid \alpha_1,\alpha_2) = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\theta^{\alpha_1-1}(1-\theta)^{\alpha_2-1}$$

Likelihood

$$P(D \mid \theta) = \frac{\Gamma(N_1+N_2)}{\Gamma(N_1)\Gamma(N_2)}\theta^{N_1}(1-\theta)^{N_2}$$

Posterior $p(\theta \mid D,\xi) = Beta(\alpha_1+N_1,\alpha_2+N_2)$

MAP estimate $\theta_{MAP} = \arg\max_{\theta}\; p(\theta \mid D,\xi)$

$$\theta_{MAP} = \frac{\alpha_1+N_1-1}{\alpha_1+\alpha_2+N_1+N_2-2}$$

Expected value of the parameter

The result is the same as for Bernoulli distribution

$$E(\theta) = \int_0^1 \theta Beta(\theta \mid \eta_1,\eta_2)d\theta = \frac{\eta_1}{\eta_1+\eta_2}$$

Expected value of the parameter

$$E(\theta) = \frac{\alpha_1+N_1}{\alpha_1+N_1+\alpha_2+N_2}$$

Predictive probability of event x=1

$$P(x=1 \mid \theta,\xi) = E(\theta) = \frac{\alpha_1+N_1}{\alpha_1+N_1+\alpha_2+N_2}$$

Multinomial distribution

Example: **Multi-way coin toss, roll of dice**

- **Data:** a set of N outcomes (multi-set)

  N_i - a number of times an outcome i has been seen

Model parameters: $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots \theta_k)$ **s.t.** $\sum_{i=1}^{k} \theta_i = 1$

θ_i - probability of an outcome i

Probability of data (likelihood)

$$P(N_1, N_2, \ldots N_k \mid \boldsymbol{\theta}, \xi) = \frac{N!}{N_1! N_2! \ldots N_k!} \theta_1^{N_1} \theta_2^{N_2} \ldots \theta_k^{N_k}$$

Multinomial distribution

ML estimate:

$$\theta_{i,ML} = \frac{N_i}{N}$$

Posterior density and MAP estimate

Choice of the prior: **Dirichlet distribution**

$$Dir(\boldsymbol{\theta} \mid \alpha_1, .., \alpha_k) = \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \theta_1^{\alpha_1 - 1} \theta_2^{\alpha_2 - 1} \ldots \theta_k^{\alpha_k - 1}$$

Dirichlet is the conjugate choice for multinomial

$$P(D \mid \boldsymbol{\theta}, \xi) = P(N_1, N_2, \ldots N_k \mid \boldsymbol{\theta}, \xi) = \frac{N!}{N_1! N_2! \ldots N_k!} \theta_1^{N_1} \theta_2^{N_2} \ldots \theta_k^{N_k}$$

Posterior density

$$p(\boldsymbol{\theta} \mid D, \xi) = \frac{P(D \mid \boldsymbol{\theta}, \xi) Dir(\boldsymbol{\theta} \mid \alpha_1, \alpha_2, .. \alpha_k)}{P(D \mid \xi)} = Dir(\boldsymbol{\theta} \mid \alpha_1 + N_1, .., \alpha_k + N_k)$$

MAP estimate:

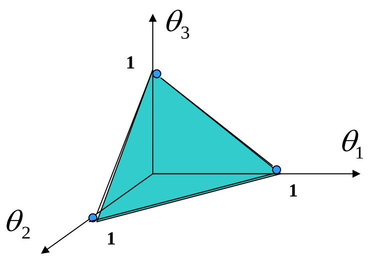
$$\theta_{i,MAP} = \frac{\alpha_i + N_i - 1}{\sum_{i=1,..k} (\alpha_i + N_i) - k}$$

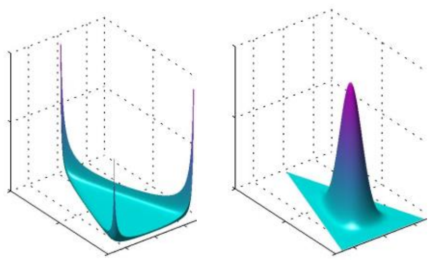
Dirichlet distribution

Dirichlet distribution:

$$Dir(\boldsymbol{\theta} \mid \alpha_1,..,\alpha_k) = \frac{\Gamma(\sum_{i=1}^{k}\alpha_i)}{\prod_{i=1}^{k}\Gamma(\alpha_i)} \theta_1^{\alpha_1-1}\theta_2^{\alpha_2-1}\ldots\theta_k^{\alpha_k-1}$$

Assume: k=3





$\alpha_k = 10^{-1}$ $\alpha_k = 10^{1}$

Expected value

The result is analogous to the result for binomial

$$E(\boldsymbol{\theta}) = \int_{0\le\theta_i\le 1, \sum\theta_i=1} \boldsymbol{\theta} \; Dir(\boldsymbol{\theta} \mid \boldsymbol{\eta}) d\boldsymbol{\theta} = \left(\frac{\eta_1}{\eta_1+\eta_2+\eta_k}, \ldots \frac{\eta_i}{\eta_1+\eta_2+\eta_k}, \ldots \frac{\eta_k}{\eta_1+\eta_2+\eta_k} \right)$$

Expectation based parameter estimate

$$E(\boldsymbol{\theta}) = \left(\frac{\alpha_1+N_1}{\alpha_1+N_1+\ldots+\alpha_k+N_k} \ldots \frac{\alpha_i+N_i}{\alpha_1+N_1+\ldots+\alpha_k+N_k} \ldots \frac{\alpha_k+N_k}{\alpha_1+N_1+\ldots+\alpha_k+N_k} \right)$$

Represents the predictive probability of an event x=i

$$P(x=i \mid \boldsymbol{\theta}, \xi) = \frac{\alpha_i+N_i}{\alpha_1+N_1+\ldots+\alpha_k+N_k}$$

Other distributions

The same ideas can be applied to other distributions

– Typically we choose distributions that behave well so that computations lead to a nice solutions

- **Exponential family of distributions**

Conjugate choices for some of the distributions from the exponential family:

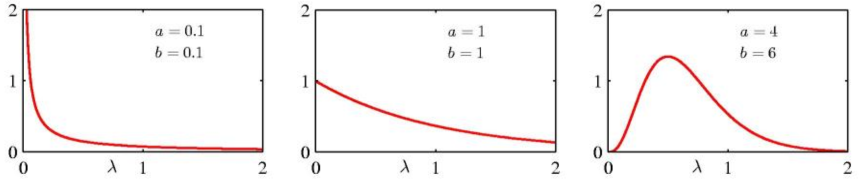
– **Binomial – Beta**
– **Multinomial - Dirichlet**
– **Exponential – Gamma**
– **Poisson – Inverse Gamma**
– **Gaussian - Gaussian (mean) and Wishart (covariance)**

Gamma distribution

- **Gamma distribution**

$$\mathrm{Gam}(\lambda|a,b) = \frac{1}{\Gamma(a)} b^a \lambda^{a-1} \exp(-b\lambda)$$

$$\mathbb{E}[\lambda] = \frac{a}{b} \qquad \mathrm{var}[\lambda] = \frac{a}{b^2}$$



Other distributions

Exponential distribution:

- A special case of Gamma for a=1

$$p(x \mid b) = \left(\frac{1}{b}\right) e^{-\frac{x}{b}}$$

Poisson distribution:

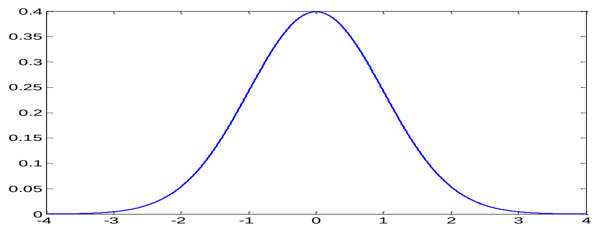
$$p(x \mid \lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!} \qquad \text{for} \quad x \in \{0,1,2,\ldots\}$$

Gaussian (normal) distribution

- **Gaussian:** $x \sim N(\mu, \sigma)$
- **Parameters:** μ - mean
  σ - standard deviation
- **Density function:**

$$p(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]$$

- **Example:**



$N(0,1)$

Parameter estimates

- **Loglikelihood** $$l(D,\mu,\sigma) = \log \prod_{i=1}^{n} p(x_i \mid \mu,\sigma)$$
- **ML estimates of the mean and variance:**

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

  – ML variance estimate is biased

$$E_n(\sigma^2) = E_n\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2\right) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

- **Unbiased estimate:**

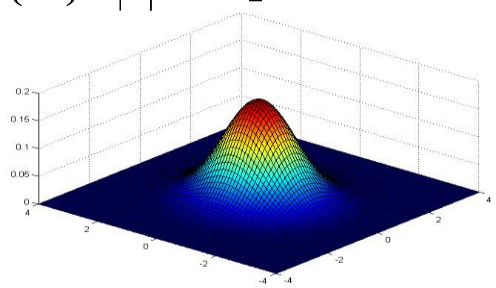
$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

Multivariate normal distribution

- **Multivariate normal:** $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- **Parameters:** $\boldsymbol{\mu}$ - mean
  $\boldsymbol{\Sigma}$ - covariance matrix
- **Density function:**

$$p(\mathbf{x} \mid \boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

- **Example:**



Partitioned Gaussian Distributions

- **Multivariate Gaussian:**

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

  – **what are marginals and conditionals?**

- **Example:**

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}$$

$$\boldsymbol{\Lambda} \equiv \boldsymbol{\Sigma}^{-1} \qquad \boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}$$

Precision matrix

Partitioned Conditionals and Marginals

- **Conditional density:**

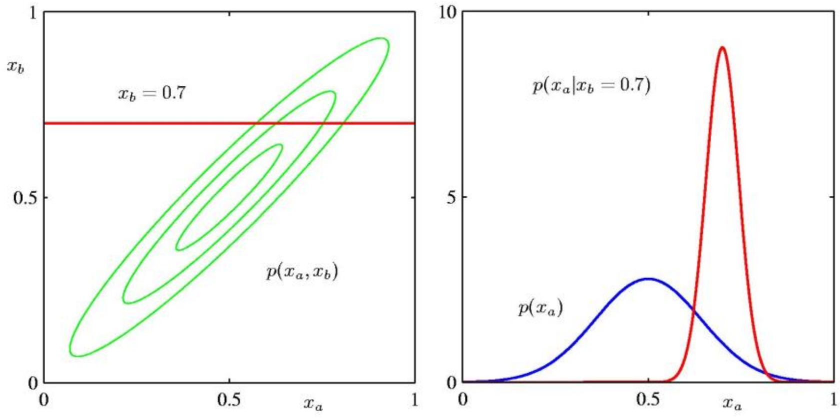
$$p(\mathbf{x}_a|\mathbf{x}_b) = \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_{a|b}, \boldsymbol{\Sigma}_{a|b})$$

$$\begin{aligned} \boldsymbol{\Sigma}_{a|b} &= \boldsymbol{\Lambda}_{aa}^{-1} = \boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\boldsymbol{\Sigma}_{ba} \\ \boldsymbol{\mu}_{a|b} &= \boldsymbol{\Sigma}_{a|b}\left\{\boldsymbol{\Lambda}_{aa}\boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b)\right\} \\ &= \boldsymbol{\mu}_a - \boldsymbol{\Lambda}_{aa}^{-1}\boldsymbol{\Lambda}_{ab}(\mathbf{x}_b - \boldsymbol{\mu}_b) \\ &= \boldsymbol{\mu}_a + \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}(\mathbf{x}_b - \boldsymbol{\mu}_b) \end{aligned}$$

- **Marginal Density:**

$$\begin{aligned} p(\mathbf{x}_a) &= \int p(\mathbf{x}_a, \mathbf{x}_b)\, \mathrm{d}\mathbf{x}_b \\ &= \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa}) \end{aligned}$$

Partitioned Conditionals and Marginals



Parameter estimates

- **Loglikelihood**

$$l(D, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \log \prod_{i=1}^{n} p(\mathbf{x}_i \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- **ML estimates of the mean and covariances:**

$$\hat{\boldsymbol{\mu}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T$$

  – Covariance estimate is biased

$$E_n(\hat{\boldsymbol{\Sigma}}) = E_n\left(\frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T \right) = \frac{n-1}{n} \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}$$

- **Unbiased estimate:**

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T$$

Posterior of a multivariate normal

- **Assume a prior on the mean $\boldsymbol{\mu}$ that is normally distributed:**

$$p(\boldsymbol{\mu}) \approx N(\boldsymbol{\mu}_p, \boldsymbol{\Sigma}_p)$$

- **Then the posterior of $\boldsymbol{\mu}$ is normally distributed**

$$p(\boldsymbol{\mu} \mid D) \approx \left(\prod_{i=1}^{n} \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) \right] \right)$$

$$* \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}_p|^{1/2}} \exp\left[-\frac{1}{2} (\boldsymbol{\mu} - \boldsymbol{\mu}_p)^T \boldsymbol{\Sigma}_p^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_p) \right]$$

$$= \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}_n|^{1/2}} \exp\left[-\frac{1}{2} (\boldsymbol{\mu} - \boldsymbol{\mu}_n)^T \boldsymbol{\Sigma}_n^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_n) \right]$$

Posterior of a multivariate normal

- **Then the posterior of $\boldsymbol{\mu}$ is normally distributed**

$$p(\boldsymbol{\mu} \mid D) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}_n|^{1/2}} \exp\left[-\frac{1}{2} (\boldsymbol{\mu} - \boldsymbol{\mu}_n)^T \boldsymbol{\Sigma}_n^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_n) \right]$$

$$\boldsymbol{\Sigma}_n^{-1} = n\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma}_p^{-1}$$

$$\boldsymbol{\mu}_n = \boldsymbol{\Sigma}_p \left(\boldsymbol{\Sigma}_p + \frac{1}{n} \boldsymbol{\Sigma} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) + \frac{1}{n} \boldsymbol{\Sigma} \left(\boldsymbol{\Sigma}_p + \frac{1}{n} \boldsymbol{\Sigma} \right)^{-1} \boldsymbol{\mu}_p$$

$$\boldsymbol{\Sigma}_n = \boldsymbol{\Sigma}_p \left(\boldsymbol{\Sigma}_p + \frac{1}{n} \boldsymbol{\Sigma} \right)^{-1} \frac{1}{n} \boldsymbol{\Sigma}$$

Sequential Bayesian parameter estimation

- **Sequential Bayesian approach**
  - Under the iid the estimates of the posterior can be computed incrementally for a sequence of data points

$$p(\Theta \mid D, \xi) = \frac{p(D \mid \Theta, \xi)\, p(\Theta \mid \xi)}{\int_{\Theta} p(D \mid \Theta, \xi)\, p(\Theta \mid \xi) d\Theta}$$

- If we use a conjugate prior we get back the same posterior
- Assume we split the data D in the last element **x** and the rest
  $p(D \mid \mathbf{\Theta}) = P(x \mid \mathbf{\Theta}) P(D_{n-1} \mid \mathbf{\Theta})$
- **Then:**

$$p(\Theta \mid D, \xi) = \frac{P(x \mid \mathbf{\Theta}) \overbrace{P(D_{n-1} \mid \mathbf{\Theta})\, p(\Theta \mid \xi)}^{\textbf{A “new” prior}}}{\int_{\Theta} P(x \mid \mathbf{\Theta}) P(D_{n-1} \mid \mathbf{\Theta})\, p(\Theta \mid \xi) d\Theta}$$