CS 2750 Machine Learning
Lecture 11

Multilayer neural networks

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Midterm exam

Midterm Monday, March 2, 2015

- **In-class (75 minutes)**
- **closed book**
- **material covered by February 25, 2015**

Multilayer neural networks

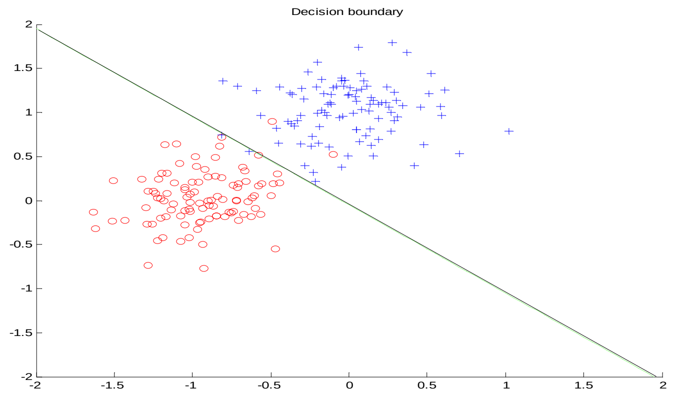
Or another way of modeling nonlinearities for regression and classification problems

CS 2750 Machine Learning

Classification with the linear model.

Logistic regression model defines a linear decision boundary

- Example: 2 classes (blue and red points)



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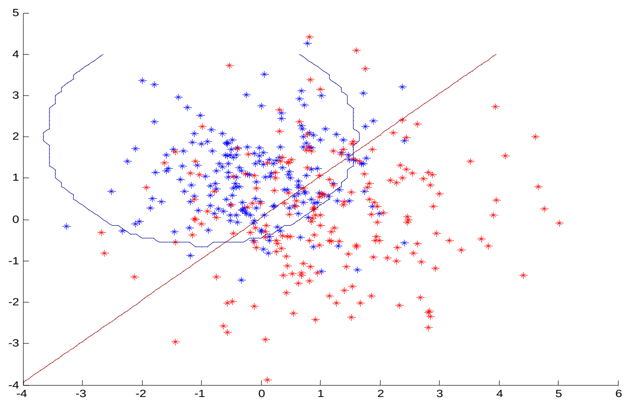
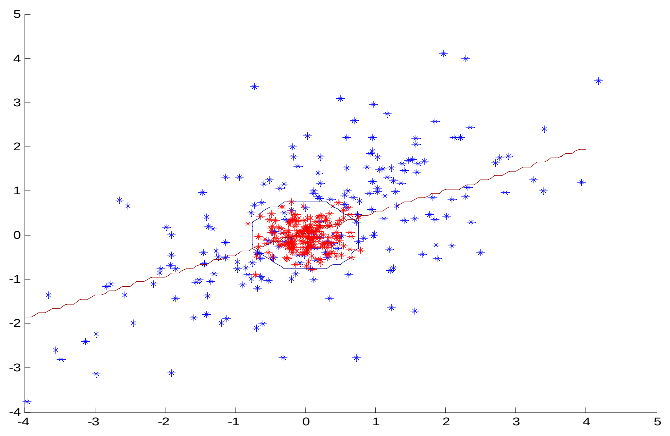
Linear decision boundary

- logistic regression model is not optimal, but not that bad



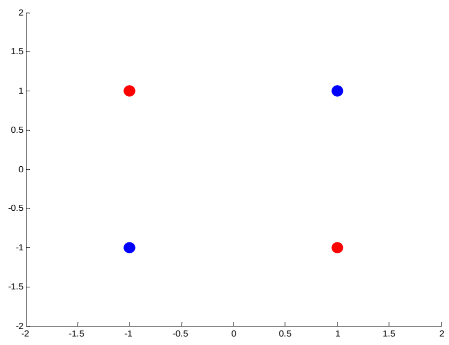
When logistic regression fails?

- Example in which the logistic regression model fails



Limitations of linear units.

- Logistic regression does not work for **parity functions**
  - no linear decision boundary exists



Solution: a model of a non-linear decision boundary

Extensions of simple linear units

- use **feature (basis) functions** to model **nonlinearities**

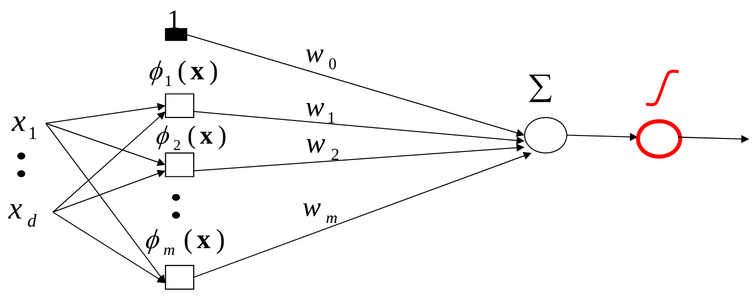
Linear regression

$$f(\mathbf{x}) = w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x})$$

Logistic regression

$$f(\mathbf{x}) = g(w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x}))$$

$\phi_j(\mathbf{x})$ - an arbitrary function of $\mathbf{x}$



Learning with extended linear units

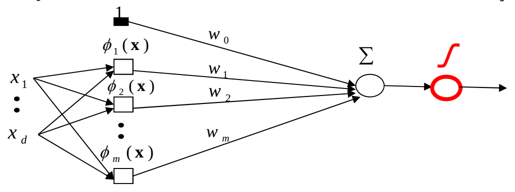
Feature (basis) functions model **nonlinearities**

Linear regression

$$f(\mathbf{x}) = w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x})$$

Logistic regression

$$f(\mathbf{x}) = g(w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x}))$$



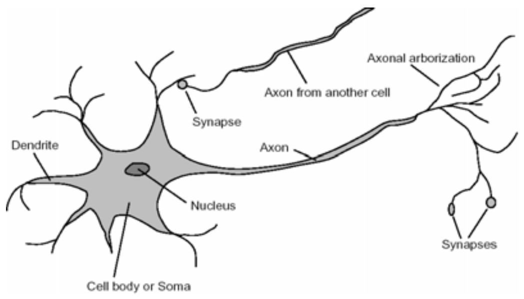
Important property:

- The same problem as learning of the weights for linear units , the input has changed– but the weights are linear in the new input

Problem: too many weights to learn

Multi-layered neural networks

- An alternative way to introduce **nonlinearities to regression/classification models**
- **Key idea: Cascade several simple neural models with logistic units.** Much like neuron connections.

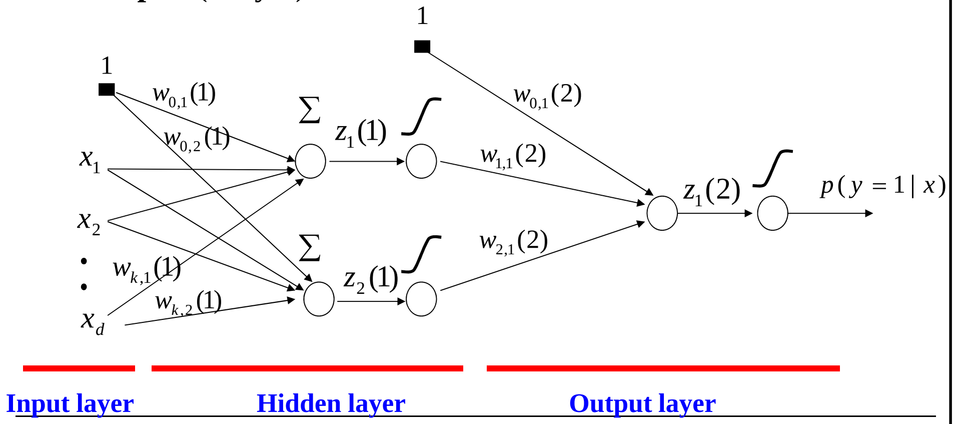


Multilayer neural network

Also called a **multilayer perceptron (MLP)**

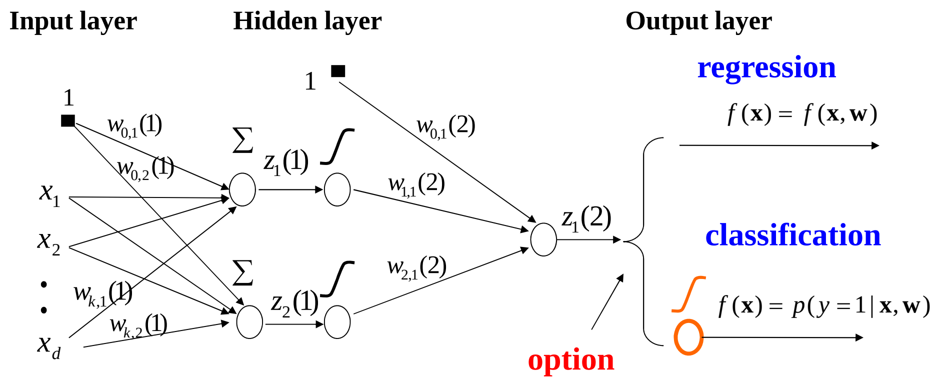
Cascades multiple logistic regression units

Example: (2 layer) classifier with non-linear decision boundaries



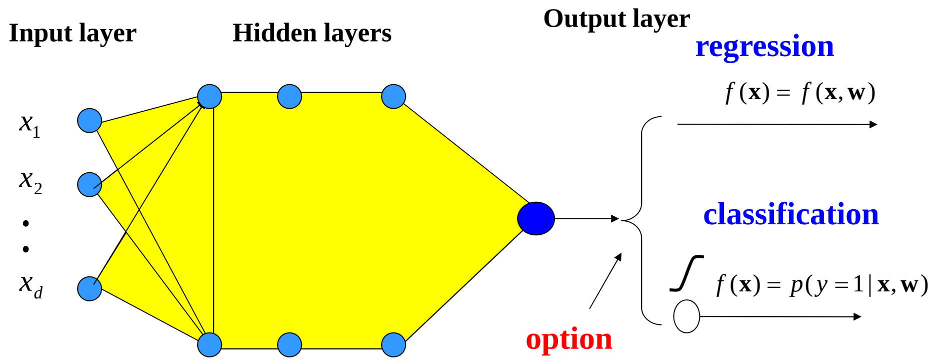
Multilayer neural network

- Models **non-linearity through logistic regression units**
- Can be applied to both **regression and binary classification problems**



Multilayer neural network

- **Non-linearities are modeled using multiple hidden logistic regression units (organized in layers)**
- The output layer determines whether it is a **regression or a binary classification problem**

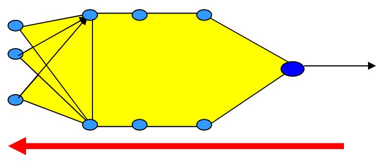

Input layer
Hidden layers
Output layer
regression
$f(\mathbf{x}) = f(\mathbf{x}, \mathbf{w})$
classification
$f(\mathbf{x}) = p(y=1 \mid \mathbf{x}, \mathbf{w})$
x_1
x_2
x_d
option

Learning with MLP

- How to learn the parameters of the neural network?
- **Gradient descent algorithm**
  - Weight updates based on the error: $J(D, \mathbf{w})$

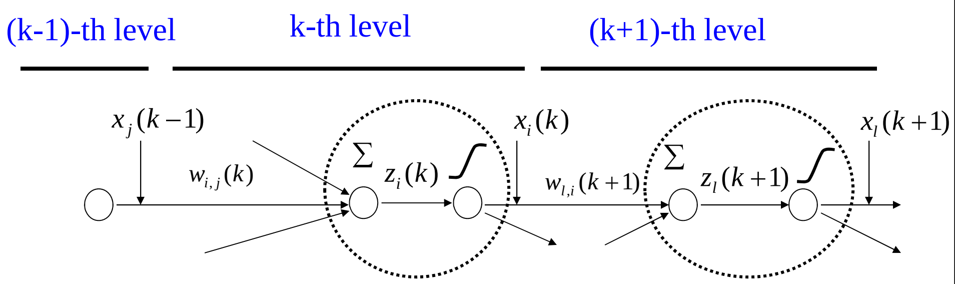
$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \nabla_{\mathbf{w}} J(D, \mathbf{w})$$

- We need to **compute gradients for weights in all units**
- **<u>Can be computed in one backward sweep through the net !!!</u>**



- The process is called **back-propagation**

Backpropagation



$x_i(k)$ - output of the unit i on level k

$z_i(k)$ - input to the sigmoid function on level k

$w_{i,j}(k)$ - weight between units j and i on levels (k-1) and k

$$z_i(k) = w_{i,0}(k) + \sum_j w_{i,j}(k) x_j(k-1)$$

$$x_i(k) = g(z_i(k))$$

Backpropagation

Update weight $w_{i,j}(k)$ using a data point $D = \{<\mathbf{x}, y>\}$

$$w_{i,j}(k) \leftarrow w_{i,j}(k) - \alpha \frac{\partial}{\partial w_{i,j}(k)} J(D, \mathbf{w})$$

Let $\delta_i(k) = \frac{\partial}{\partial z_i(k)} J(D, \mathbf{w})$

Then: $$\frac{\partial}{\partial w_{i,j}(k)} J(D, \mathbf{w}) = \frac{\partial J(D, \mathbf{w})}{\partial z_i(k)} \frac{\partial z_i(k)}{\partial w_{i,j}(k)} = \delta_i(k) x_j(k-1)$$

S.t. $\delta_i(k)$ is computed from $x_i(k)$ and the next layer $\delta_l(k+1)$

$$\delta_i(k) = \left[\sum_l \delta_l(k+1) w_{l,i}(k+1) \right] x_i(k)(1 - x_i(k))$$

Last unit (is the same as for the regular linear units):

$$\delta_i(K) = -(y_u - f(\mathbf{x}_u, \mathbf{w}))$$

It is the same for the classification with the log-likelihood measure of fit and linear regression with least-squares error!!!

Learning with MLP

- **Online gradient descent algorithm**
  - Weight update:

$$w_{i,j}(k) \leftarrow w_{i,j}(k) - \alpha \frac{\partial}{\partial w_{i,j}(k)} J_{\text{online}}(D_u, \mathbf{w})$$

$$\frac{\partial}{\partial w_{i,j}(k)} J_{online}(D_u, \mathbf{w}) = \frac{\partial J_{online}(D_u, \mathbf{w})}{\partial z_i(k)} \frac{\partial z_i(k)}{\partial w_{i,j}(k)} = \delta_i(k) x_j(k-1)$$

$$\boxed{w_{i,j}(k) \leftarrow w_{i,j}(k) - \alpha \delta_i(k) x_j(k-1)}$$

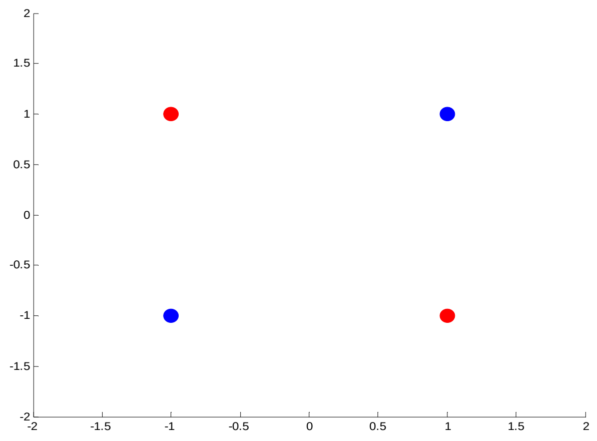
$x_j(k-1)$ - j-th output of the (k-1) layer
$\delta_i(k)$ - derivative computed via backpropagation
α - a learning rate

Online gradient descent algorithm for MLP

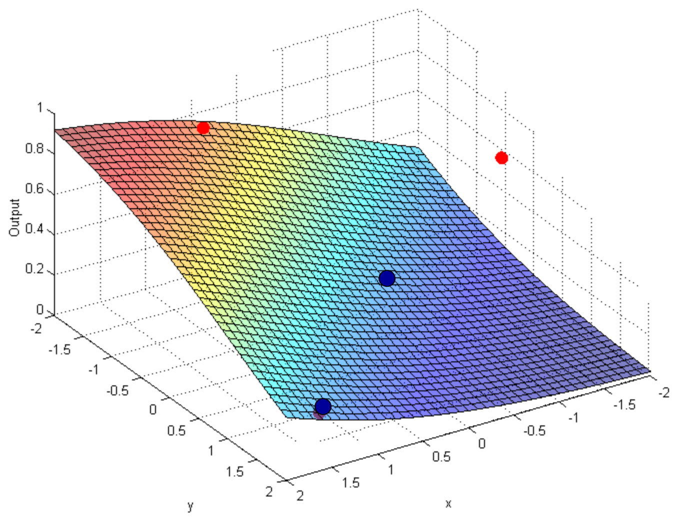
```
Online-gradient-descent (D, number of iterations)
    Initialize all weights $w_{i,j}(k)$
    for i=1:1: number of iterations
       do     select a data point $D_u$=<x,y> from D
               set learning rate $\alpha$
               compute outputs $x_j(k)$ for each unit
               compute derivatives $\delta_i(k)$ via backpropagation
               update all weights (in parallel)
                  $w_{i,j}(k) \leftarrow w_{i,j}(k) - \alpha \delta_i(k) x_j(k-1)$
    end for
    return weights w
```

Xor Example.

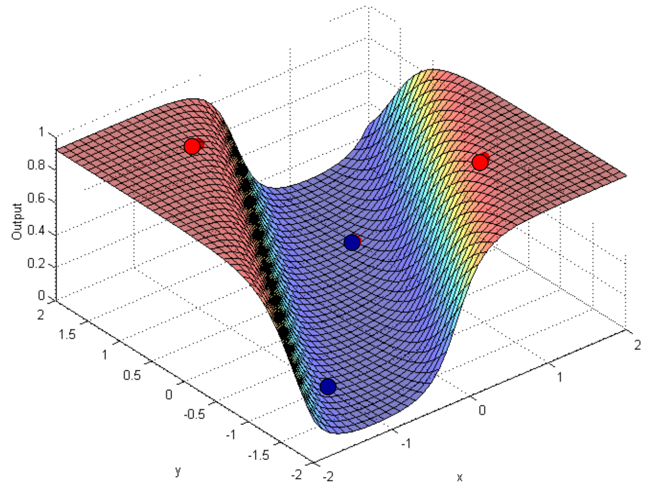
- linear decision boundary does not exist



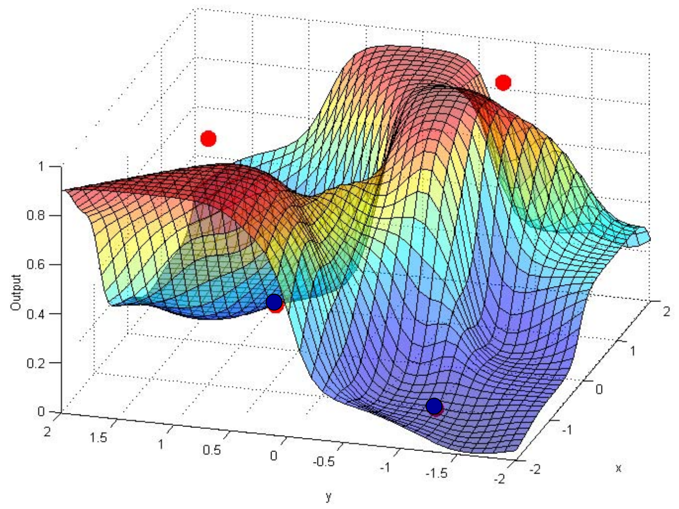
Xor example. Linear unit



Xor example.
Neural network with 2 hidden units

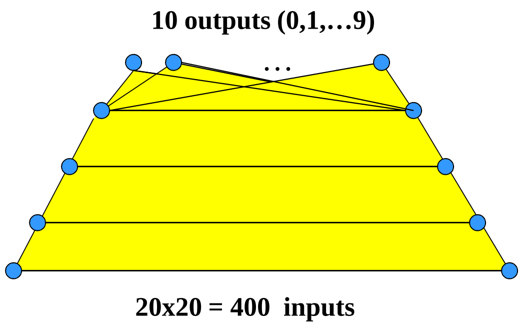


Xor example.
Neural network with 10 hidden units



MLP in practice

- **Optical character recognition** – digits 20x20
  - Automatic sorting of mails
  - 5 layer network with multiple output functions



layer	Neurons	Weights
5	10	3000
4	300	1200
3	1200	50000
2	784	3136
1	3136	78400