CS 2750 Machine Learning
Lecture 17

Expectation Maximization (EM).
Mixtures of Gaussians.

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Learning probability distribution

Basic learning settings:

- A set of random variables $\mathbf{X} = \{X_1, X_2, \dots, X_n\}$
- **A model of the distribution** over variables in ***X*** with parameters Θ
- **Data** $D = \{D_1, D_2, .., D_N\}$

  s.t. $D_i = (x_1^i, x_2^i, \dots x_n^i)$

Objective: find parameters $\hat{\Theta}$ that describe the data

Assumptions considered so far:

- Known parameterizations
- No hidden variables
- No-missing values

General EM

The key idea of a method:

Compute the parameter estimates iteratively by performing the following two steps:

Two steps of the EM:

1. **Expectation step**. Complete all hidden and missing variables with expectations for the current set of parameters $\boldsymbol{\Theta}'$
2. **Maximization step**. Compute the new estimates of $\boldsymbol{\Theta}$ for the completed data

Stop when no improvement possible

EM

Let H – be a set of hidden or missing values

Derivation

$$P(H, D \mid \Theta, \xi) = P(H \mid D, \Theta, \xi) P(D \mid \Theta, \xi)$$

$$\log P(H, D \mid \Theta, \xi) = \log P(H \mid D, \Theta, \xi) + \log P(D \mid \Theta, \xi)$$

$$\underline{\log P(D \mid \Theta, \xi)} = \log P(H, D \mid \Theta, \xi) - \log P(H \mid D, \Theta, \xi)$$

Log-likelihood of data

Average both sides with $P(H \mid D, \Theta', \xi)$ **for some** Θ'

$$E_{H|D,\Theta'} \log P(D \mid \Theta, \xi) = E_{H|D,\Theta'} \log P(H, D \mid \Theta, \xi) - E_{H|D,\Theta'} \log P(H \mid \Theta, \xi)$$

$$\underbrace{\log P(D \mid \Theta, \xi)}_{\text{Log-likelihood of data}} = Q(\Theta \mid \Theta') + H(\Theta \mid \Theta')$$

Log-likelihood of data

EM algorithm

Algorithm (general formulation)

Initialize parameters Θ

Repeat

Set $\Theta' = \Theta$

1. **Expectation step**

$$Q(\Theta \mid \Theta') = E_{H|D,\Theta'} \log P(H, D \mid \Theta, \xi)$$

2. **Maximization step**

$$\Theta = \arg\max_{\Theta} Q(\Theta \mid \Theta')$$

until no or small improvement in Θ $(\Theta = \Theta')$

Questions: Why this leads to the ML estimate ?

What is the advantage of the algorithm?

EM algorithm

- Why is the EM algorithm correct?
- **Claim: maximizing Q improves the log-likelihood**

$$l(\Theta) = Q(\Theta \mid \Theta') + H(\Theta \mid \Theta')$$

Difference in log-likelihoods (current and next step)

$$l(\Theta) - l(\Theta') = Q(\Theta \mid \Theta') - Q(\Theta' \mid \Theta') + H(\Theta \mid \Theta') - H(\Theta' \mid \Theta')$$

Subexpression $H(\Theta \mid \Theta') - H(\Theta' \mid \Theta') \geq 0$

Kullback-Leibler (KL) divergence (distance between 2 distributions)

$$KL(P \mid R) = \sum_i P_i \log \frac{P_i}{R_i} \geq 0$$ Is always positive !!!

$$H(\Theta \mid \Theta') = -E_{H|D,\Theta'} \log P(H \mid \Theta, D, \xi) = -\sum_i p(H \mid D, \Theta') \log P(H \mid \Theta, D, \xi)$$

$$H(\Theta \mid \Theta') - H(\Theta' \mid \Theta') = \sum_i P(H \mid D, \Theta') \log \frac{P(H \mid \Theta', D, \xi)}{P(H \mid \Theta, D, \xi)} \geq 0$$

EM algorithm

Difference in log-likelihoods

$$l(\Theta) - l(\Theta') = Q(\Theta \mid \Theta') - Q(\Theta' \mid \Theta') + H(\Theta \mid \Theta') - H(\Theta' \mid \Theta')$$

$$l(\Theta) - l(\Theta') \geq Q(\Theta \mid \Theta') - Q(\Theta' \mid \Theta')$$

Thus

by **maximizing Q we maximize the log-likelihood**

$$l(\Theta) = Q(\Theta \mid \Theta') + H(\Theta \mid \Theta')$$

EM is a first-order optimization procedure

- **Climbs the gradient**
- **Automatic learning rate**

No need to adjust the learning rate !!!!

EM advantages

Key advantages:

- In many problems (e.g. Bayesian belief networks)

$$Q(\Theta \mid \Theta') = E_{H|D,\Theta'} \log P(H, D \mid \Theta, \xi)$$

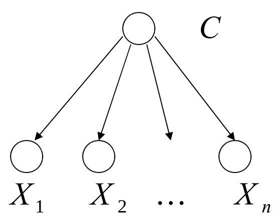
  – has a nice form and the maximization of Q can be carried out in the closed form
- No need to compute Q before maximizing
- We directly optimize
  – using quantities corresponding to expected counts

Naïve Bayes with a hidden class and missing values

Assume:

- $P(\mathbf{X})$ is modeled using a Naïve Bayes model with hidden class variable
- Missing entries (values) for attributes in the dataset D

Hidden class variable



Attributes are independent given the class

EM for the Naïve Bayes

- **We can use EM to learn the parameters**

$$Q(\Theta \mid \Theta') = E_{H|D,\Theta'} \log P(H, D \mid \Theta, \xi)$$

- **Parameters:**
  - π_j prior on class j
  - θ_{ijk} probability of an attribute i having value k given class j
- **Indicator variables:**
  - δ_j^l for example l, the class is j ; if true (=1) else false (=0)
  - δ_{ijk}^l for example l, the class is j and the value of attrib i is k
- because the class is hidden and some attributes are missing, the values (0,1) of indicator variables are not known; they are hidden

H – a collection of all indicator variables

EM for the Naïve Bayes model

- We can use EM to do the learning of parameters

$$Q(\Theta \mid \Theta') = E_{H|D,\Theta'} \log P(H, D \mid \Theta, \xi)$$

$$\log P(H, D \mid \Theta, \xi) = \log \prod_{l=1}^{N} \prod_{j} \pi_j^{\delta_j^l} \prod_i \prod_k \theta_{ijk}^{\delta_{ijk}^l}$$

$$= \sum_{l=1}^{N} \sum_{j} (\delta_j^l \log \pi_j + \sum_i \sum_k \delta_{ijk}^l \log \theta_{ijk})$$

$$E_{H|D,\Theta'} \log P(H, D \mid \Theta, \xi) = \sum_{l=1}^{N} \sum_{j} (E_{H|D,\Theta'}(\delta_j^l) \log \pi_j + \sum_i \sum_k E_{H|D,\Theta'}(\delta_{ijk}^l) \log \theta_{ijk})$$

$$E_{H|D,\Theta'}(\delta_j^l) = p(C_l = j \mid D_l, \Theta')$$

$$E_{H|D,\Theta'}(\delta_{ijk}^l) = p(X_{il} = k, C_l = j \mid D_l, \Theta')$$

Substitutes 0,1 with expected value

EM for the Naïve Bayes model

- Computing derivatives of $\boldsymbol{Q}$ for parameters and setting it to 0 we get:

$$\pi_j = \frac{\tilde{N}_j}{N} \qquad \theta_{ijk} = \frac{\tilde{N}_{ijk}}{\sum_{k=1}^{r_i} \tilde{N}_{ijk}}$$

$$\tilde{N}_j = \sum_{l=1}^{N} E_{H|D,\Theta'}(\delta_j^l) = \sum_{l=1}^{N} p(C_l = j \mid D_l, \Theta')$$

$$\tilde{N}_{ijk} = \sum_{l=1}^{N} E_{H|D,\Theta'}(\delta_{ijk}^l) = \sum_{l=1}^{N} p(X_{il} = k, C_l = j \mid D_l, \Theta')$$

- **Important:**
  - **Use expected counts instead of counts !!!**
  - Re-estimate the parameters using expected counts

EM for BBNs

- The same result applies to learning of parameters of **any Bayesian belief network** with discrete-valued variables

$$Q(\Theta \mid \Theta') = E_{H|D,\Theta'} \log P(H, D \mid \Theta, \xi)$$

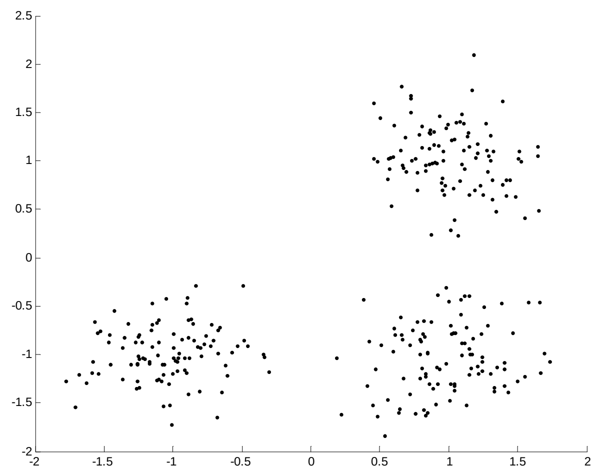
$$\theta_{ijk} = \frac{\tilde{N}_{ijk}}{\sum_{k=1}^{r_i} \tilde{N}_{ijk}}$$ ← Parameter value maximizing ***Q***

$$\tilde{N}_{ijk} = \sum_{l=1}^{N} p(x_i^l = k, pa_i^l = j \mid D^l, \Theta')$$

may require inference

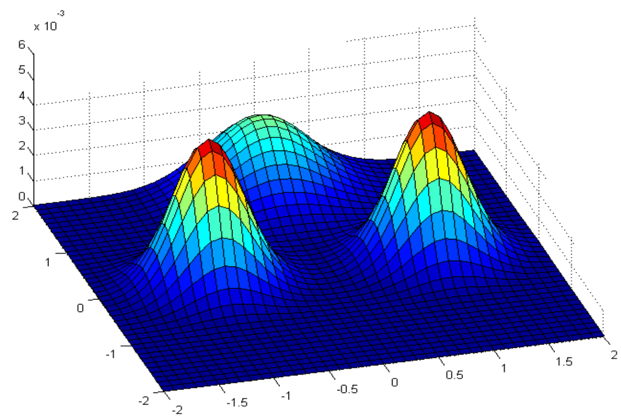
- **Again:**
  - Use expected counts instead of counts

Gaussian mixture model



Mixture of Gaussians

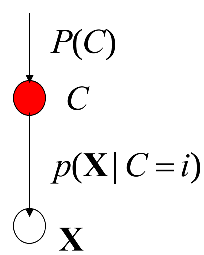
- Density function for the Mixture of Gaussians model



Gaussian mixture model

Probability of occurrence of a data point $\boldsymbol{x}$ is modeled as

$$p(\mathbf{x}) = \sum_{i=1}^{k} p(C=i)\, p(\mathbf{x} \mid C=i)$$



where

$p(C=i)$

= probability of a data point coming from class $C=i$

$p(\mathbf{x} \mid C=i) \approx N(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$

= class conditional density (modeled as a Gaussian) for class I

Special feature: ***C* is hidden !!!!**

Generative Naïve Bayes classifier model

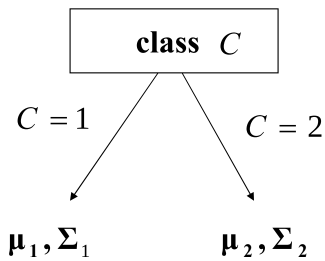
- Generative classifier model based on the Naïve Bayes
- Assume the class labels are known. The ML estimate is

$$N_i = \sum_{j: C_l = i} 1$$

$$\tilde{\pi}_i = \frac{N_i}{N}$$

$$\tilde{\boldsymbol{\mu}}_i = \frac{1}{N_i} \sum_{j: C_l = i} \mathbf{x}_j$$

$$\tilde{\boldsymbol{\Sigma}}_i = \frac{1}{N_i} \sum_{j: C_l = i} (\mathbf{x}_j - \boldsymbol{\mu}_i)(\mathbf{x}_j - \boldsymbol{\mu}_i)^T$$



Gaussian mixture model

- In the Gaussian mixture Gaussians are not labeled
- We can apply **EM algorithm**:
  - re-estimation based on the class posterior

$$h_{il} = p(C_l = i \mid \mathbf{x}_l, \Theta') = \frac{p(C_l = i \mid \Theta') p(x_l \mid C_l = i, \Theta')}{\sum_{u=1}^{m} p(C_l = u \mid \Theta') p(x_l \mid C_l = u, \Theta')}$$

$$N_i = \sum_l h_{il}$$

Count replaced with the expected count

$$\tilde{\pi}_i = \frac{N_i}{N}$$

$$\tilde{\boldsymbol{\mu}}_i = \frac{1}{N_i} \sum_l h_{il} \mathbf{x}_j$$

$$\tilde{\boldsymbol{\Sigma}}_i = \frac{1}{N_i} \sum_l h_{il} (\mathbf{x}_j - \boldsymbol{\mu}_i)(\mathbf{x}_j - \boldsymbol{\mu}_i)^T$$

Gaussian mixture algorithm

- **Special case:** fixed covariance matrix for all hidden groups (classes) and uniform prior on classes
- **Algorithm:**

  Initialize means $\boldsymbol{\mu}_i$ for all classes i

  Repeat two steps until no change in the means:

  1. Compute the class posterior for each Gaussian and each point (a kind of responsibility for a Gaussian for a point)

     Responsibility: $$h_{il} = \frac{p(C_l = i \mid \Theta')\, p(x_l \mid C_l = i, \Theta')}{\sum_{u=1}^{m} p(C_l = u \mid \Theta')\, p(x_l \mid C_l = u, \Theta')}$$

  2. Move the means of the Gaussians to the center of the data, weighted by the responsibilities

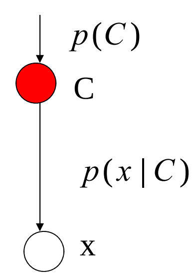
     New mean: $$\boldsymbol{\mu}_i = \frac{\sum_{l=1}^{N} h_{il} \mathbf{x}_l}{\sum_{l=1}^{N} h_{il}}$$

Gaussian mixture model. Gradient ascent.

- A set of parameters

  $$\Theta = \{\pi_1, \pi_2, .. \pi_m, \mu_1, \mu_2, ... \mu_m\}$$

  Assume unit variance terms and fixed priors



$$P(\mathbf{x} \mid C = i) = (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}\|x - \mu_i\|^2\right\}$$

$$P(D \mid \Theta) = \prod_{l=1}^{N} \sum_{i=1}^{m} \pi_i (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}\|x_l - \mu_i\|^2\right\}$$

$$l(\Theta) = \sum_{l=1}^{N} \log \sum_{i=1}^{m} \pi_i (2\pi)^{-1/2} \exp\left\{-\frac{1}{2}\|x_l - \mu_i\|^2\right\}$$

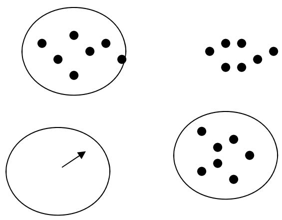
$$\frac{\partial l(\Theta)}{\partial \mu_i} = \sum_{l=1}^{N} h_{il} (x_l - \mu_i)$$ **- very easy on-line update**

EM versus gradient ascent

Gradient ascent

$$\mu_i \leftarrow \mu_i + \alpha \sum_{l=1}^{N} h_{il}(x_l - \mu_i)$$

Learning rate

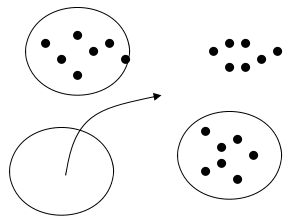


Small pull towards distant uncovered data

EM

$$\mu_i \leftarrow \frac{\sum_{l=1}^{N} h_{il} \mathbf{x}_l}{\sum_{l=1}^{N} h_{il}}$$

No learning rate



Renormalized – big jump in the first step

K-means approximation to EM

Mixture of Gaussians with the fixed covariance matrix:

- posterior measures the responsibility of a Gaussian for every point

$$h_{il} = \frac{p(C_l = i \mid \Theta')\, p(x_l \mid C_l = i, \Theta')}{\sum_{u=1}^{m} p(C_l = u \mid \Theta')\, p(x_l \mid C_l = u, \Theta')}$$

- **Re-estimation of means:** $\boldsymbol{\mu}_i = \frac{\sum_{l=1}^{N} h_{il} \mathbf{x}_l}{\sum_{l=1}^{N} h_{il}}$
- **K- Means approximations**
- Only the closest Gaussian is made responsible for a point

  $h_{il} = 1$ If i is the closest Gaussian

  $h_{il} = 0$ Otherwise

- Results in moving the means of Gaussians to the center of the data points it covered in the previous step

K-means algorithm

K-Means algorithm:

Initialize k values of means (centers)

Repeat two steps until no change in the means:

- Partition the data according to the current means (using the similarity measure)
- Move the means to the center of the data in the current partition

- **Used frequently for clustering data**