CS 2750 Machine Learning
Lecture 9

Classification with linear models

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Generative approach to classification

Idea:

1. **Represent and learn the distribution** $p(\mathbf{x}, y)$
2. **Use it to define probabilistic discriminant functions**

   E.g. $g_o(\mathbf{x}) = p(y = 0 \mid \mathbf{x})$ $\quad g_1(\mathbf{x}) = p(y = 1 \mid \mathbf{x})$

Typical model $p(\mathbf{x}, y) = p(\mathbf{x} \mid y) p(y)$

- $p(\mathbf{x} \mid y)$ = **Class-conditional distributions (densities)**

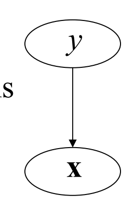
  binary classification: two class-conditional distributions

  $p(\mathbf{x} \mid y = 0)$ $\quad p(\mathbf{x} \mid y = 1)$

- $p(y)$ = **Priors on classes** **-** probability of class y

  binary classification: Bernoulli distribution

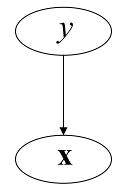
  $p(y = 0) + p(y = 1) = 1$



Generative approach to classification

Example:

- **Class-conditional distributions**
  - **multivariate normal distributions**



$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \quad \text{for} \quad y = 0$$

$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \quad \text{for} \quad y = 1$$

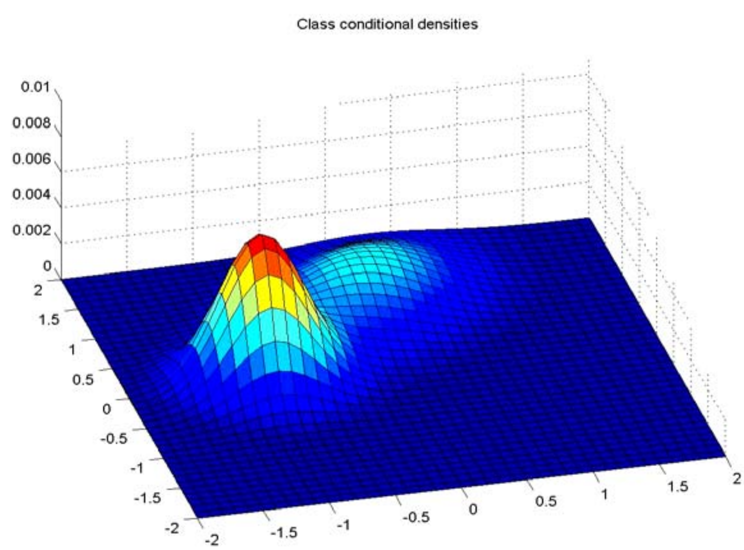
Multivariate normal $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

- **Priors on classes (class 0,1)** $y \sim Bernoulli$
  - **Bernoulli distribution**

$$p(y, \theta) = \theta^y (1 - \theta)^{1-y} \qquad y \in \{0,1\}$$

2 Gaussian class-conditional densities



Learning of parameters of the model

Density estimation problem

- We see examples & we do not know the parameters of Gaussians (class-conditional densities)

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right]$$

- **ML estimate of parameters** of a multivariate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for a set of n examples of $\mathbf{x}$

  Optimize log-likelihood: $l(D, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \log \prod_{i=1}^{n} p(\mathbf{x}_i \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\hat{\boldsymbol{\mu}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T$$

- How to learn **class priors** $p(y=0), p(y=1)$?

Making class decision

Basically we need to design **discriminant** functions

Two possible choices:

1. **Likelihood of data** – choose the class (Gaussian) that explains the input data ($\mathbf{x}$) better (likelihood of the data)

$$\underbrace{p(\mathbf{x} \mid \mu_1, \boldsymbol{\Sigma}_1)}_{g_1(\mathbf{x})} > \underbrace{p(\mathbf{x} \mid \mu_0, \boldsymbol{\Sigma}_0)}_{g_0(\mathbf{x})} \Rightarrow \text{then } y=1, \text{ else } y=0$$

2. **Posterior of a class** – choose the class with better posterior probability

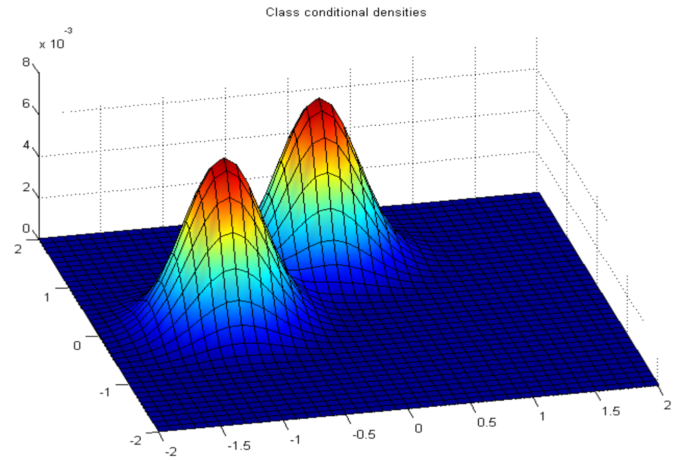
$$\underbrace{p(y=1 \mid \mathbf{x})}_{g_1(\mathbf{x})} > \underbrace{p(y=0 \mid \mathbf{x})}_{g_0(\mathbf{x})} \Rightarrow \text{then } y=1, \text{ else } y=0$$

$$p(y=1 \mid \mathbf{x}) = \frac{p(\mathbf{x} \mid \mu_1, \boldsymbol{\Sigma}_1) p(y=1)}{p(\mathbf{x} \mid \mu_0, \boldsymbol{\Sigma}_0) p(y=0) + p(\mathbf{x} \mid \mu_1, \boldsymbol{\Sigma}_1) p(y=1)}$$

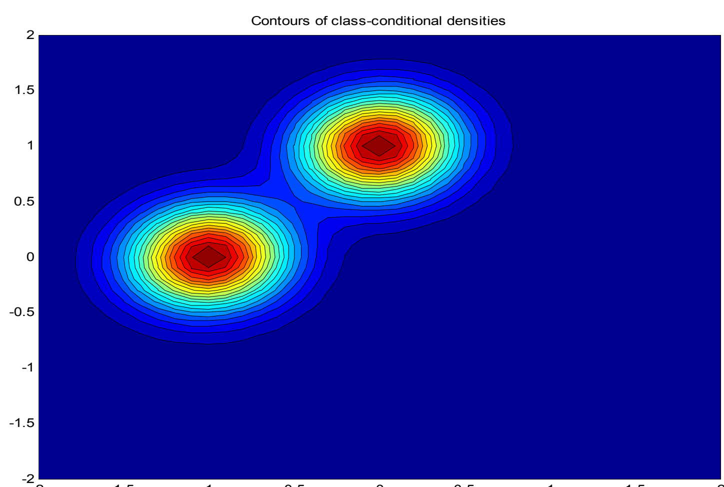
2 Gaussians: Linear decision boundary

- When covariances are the same

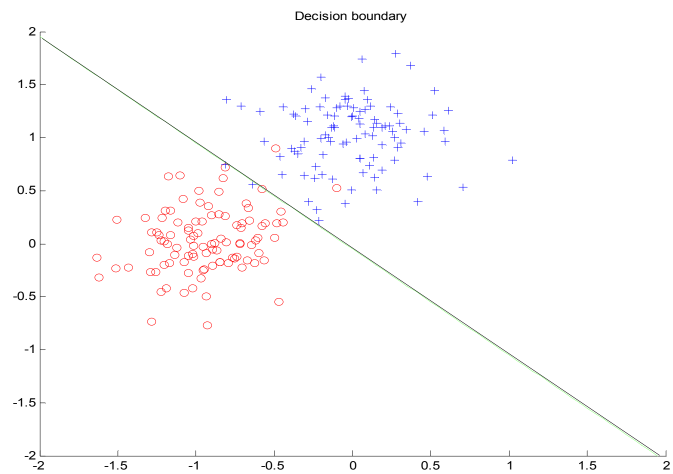
$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}),\ y = 0$$
$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}),\ y = 1$$



2 Gaussians: Linear decision boundary



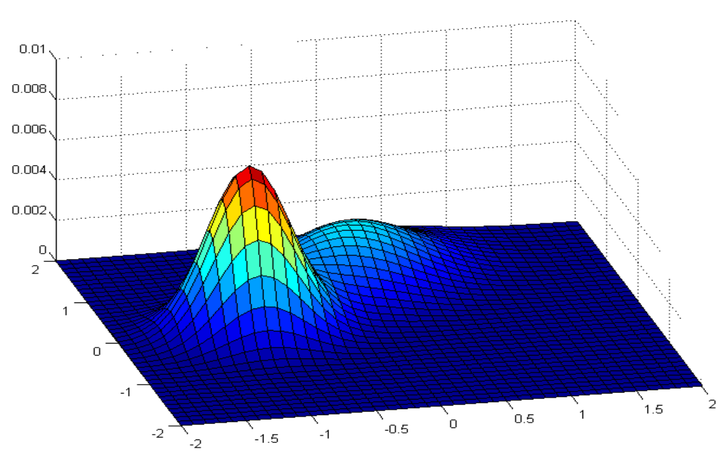
2 Gaussians: linear decision boundary



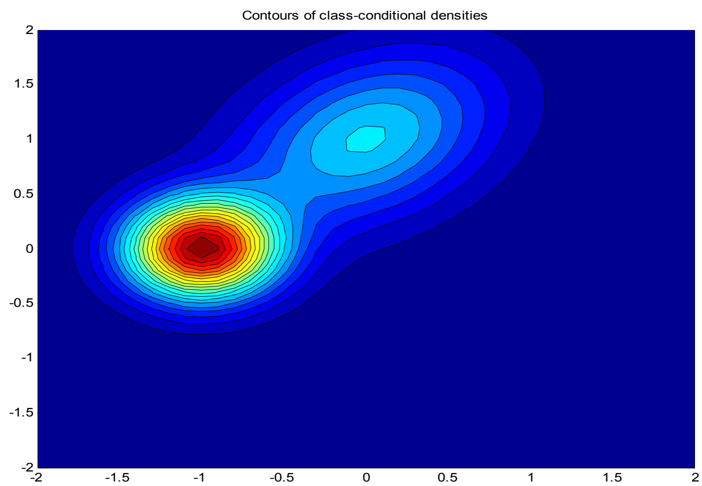
2 Gaussians: Quadratic decision boundary

- When different covariances

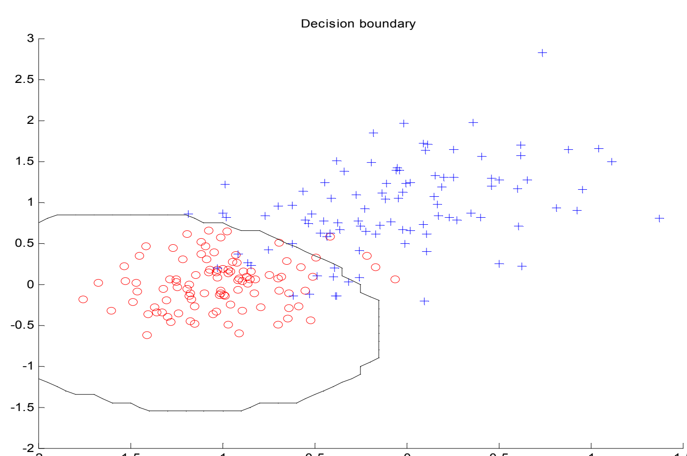
$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_1),\ y = 0$$
$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_2),\ y = 1$$



2 Gaussians: Quadratic decision boundary



2 Gaussians: Quadratic decision boundary



Back to the logistic regression

- **Two models with linear decision boundaries:**
  - **Logistic regression**
  - **Generative model with 2 Gaussians with the same covariance matrices**

$$\mathbf{x} \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \quad \text{for} \quad y = 0$$
$$\mathbf{x} \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) \quad \text{for} \quad y = 1$$

- Two models are related !!!
  - When we have **2 Gaussians with the same covariance matrices** the discriminant function has the form of a logistic regression model **!!!**

$$p(y = 1 \mid \mathbf{x}, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}) = g_1(\mathbf{w}^T \mathbf{x})$$

When is the logistic regression model correct?

- **Members of an exponential family can be often more naturally described as**

$$f(\mathbf{x} \mid \boldsymbol{\theta}, \boldsymbol{\varphi}) = h(x, \boldsymbol{\varphi}) \exp\left\{ \frac{\boldsymbol{\theta}^T \mathbf{x} - A(\boldsymbol{\theta})}{a(\boldsymbol{\varphi})} \right\}$$

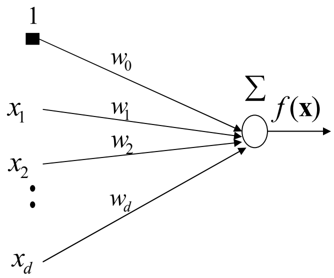
$\boldsymbol{\theta}$ - A location parameter $\quad \boldsymbol{\varphi}$ - A scale parameter

- **Claim:** A logistic regression is a correct model when class conditional densities are from the same distribution in the exponential family and have **the same scale factor** $\boldsymbol{\varphi}$
- **Very powerful result !!!!**
  - **We can represent posteriors of many distributions with the same small network**

Linear units

Linear regression

$$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$$



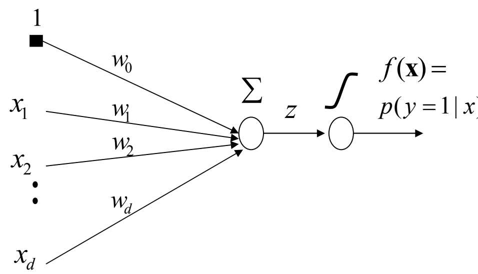
Gradient update:

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i))\mathbf{x}_i$$

Online: $\mathbf{w} \leftarrow \mathbf{w} + \alpha (y - f(\mathbf{x}))\mathbf{x}$

Logistic regression

$$f(\mathbf{x}) = p(y = 1 \mid \mathbf{x}, \mathbf{w}) = g(\mathbf{w}^T \mathbf{x})$$



Gradient update:

$$\mathbf{w} \leftarrow \mathbf{w} + \alpha \sum_{i=1}^{n} (y_i - f(\mathbf{x}_i))\mathbf{x}_i$$

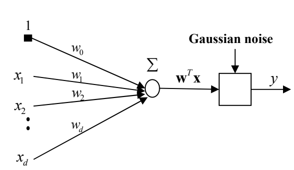
Online: $\mathbf{w} \leftarrow \mathbf{w} + \alpha (y - f(\mathbf{x}))\mathbf{x}$

The same

Gradient-based learning

- The **same simple gradient update rule** derived for both the linear and logistic regression models
- Where the magic comes from?
- Under the **log-likelihood** measure the function models and the models for the output selection fit together:
  - **Linear model + Gaussian noise**

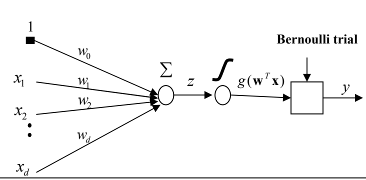
    $$y = \mathbf{w}^T \mathbf{x} + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2)$$



  - **Logistic + Bernoulli**

    $$y = \text{Bernoulli}(\theta)$$

    $$\theta = p(y = 1 \mid \mathbf{x}) = g(\mathbf{w}^T \mathbf{x})$$



Generalized linear models (GLIM)

Assumptions:

- The conditional mean (expectation) is:

$$\mu = f(\mathbf{w}^T \mathbf{x})$$

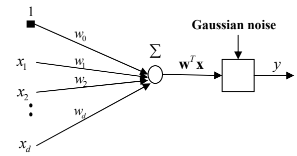
  – Where $f(.)$ is a **response function**

- Output y is characterized by an exponential family distribution with a conditional mean μ

Examples:

– **Linear model + Gaussian noise**

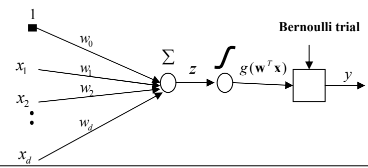
$$y = \mathbf{w}^T \mathbf{x} + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2)$$



– **Logistic + Bernoulli**

$$y \approx \text{Bernoulli}(\theta)$$

$$\theta = g(\mathbf{w}^T \mathbf{x}) = \frac{1}{1 + e^{-\mathbf{w}^T \mathbf{x}}}$$



Generalized linear models

- **A canonical response functions $f(.)$:**
  – **encoded in the distribution**

$$p(\mathbf{x} \mid \boldsymbol{\theta}, \boldsymbol{\varphi}) = h(x, \boldsymbol{\varphi}) \exp\left\{ \frac{\boldsymbol{\theta}^T \mathbf{x} - A(\boldsymbol{\theta})}{a(\boldsymbol{\varphi})} \right\}$$

- **Leads to a simple gradient form**
- **Example: Bernoulli distribution**

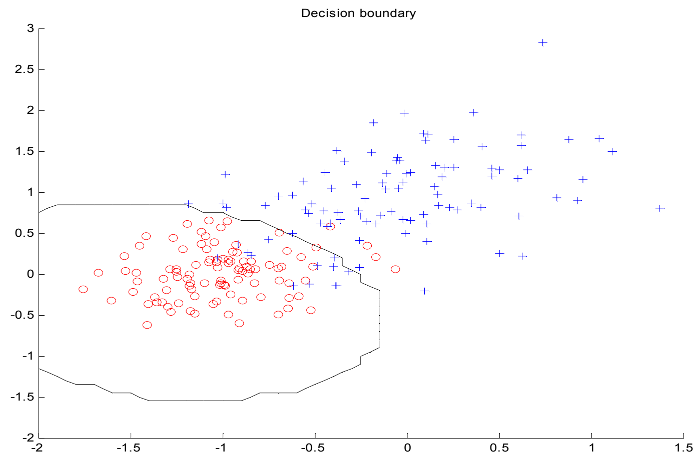
$$p(x \mid \mu) = \mu^x (1-\mu)^{1-x} = \exp\left\{ \log\left(\frac{\mu}{1-\mu} \right) x + \log(1-\mu) \right\}$$

$$\theta = \log\left(\frac{\mu}{1-\mu} \right) \qquad \mu = \frac{1}{1+e^{-\theta}}$$

  – **Logistic function matches the Bernoulli**

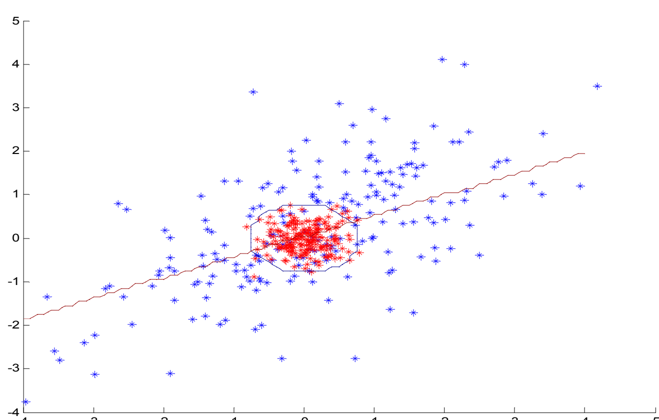
When does the logistic regression fail?

- Quadratic decision boundary is needed



When does the logistic regression fail?

- Another example of a non-linear decision boundary



Non-linear extension of logistic regression

- use **feature (basis) functions** to model **nonlinearities**
  - the same trick as used for the linear regression

Linear regression

$$f(\mathbf{x}) = w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x})$$

Logistic regression

$$f(\mathbf{x}) = g(w_0 + \sum_{j=1}^{m} w_j \phi_j(\mathbf{x}))$$

$\phi_j(\mathbf{x})$ - an arbitrary function of $\mathbf{x}$