CS 2710 Foundations of AI
Lecture 11

First-order logic

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Administration

- PS-5 is out

Midterm:
- October 24, 2005
- In class
- Closed book
- Covers:
  Search, Knowledge Representation and Planning

Limitations of propositional logic

World we want to represent and reason about consists of a number of objects with variety of properties and relations among them

Propositional logic:

- Represents statements about the world without reflecting this structure and without modeling these entities explicitly

Consequence:

- some knowledge is hard or impossible to encode in the propositional logic.
- Two cases that are hard to represent:
  - **Statements about similar objects, relations**
  - **Statements referring to groups of objects**.

First-order logic (FOL)

- More expressive than **propositional logic**

- **Eliminates deficiencies of PL by:**
  - Representing objects, their properties, relations and statements about them;
  - Introducing variables that refer to an arbitrary objects and can be substituted by a specific object
  - Introducing quantifiers allowing us to make statements over groups objects without the need to represent each of them separately

Logic

Logic is defined by**:**

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with objects, values in the real world.
- **The valuation (meaning) function** V
  - Assigns a truth value to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

First-order logic. Syntax.

Term - syntactic entity for representing objects

Terms in FOL:

- **Constant symbols:** represent specific objects
  Examples: *John, France, car89*

- **Variables** – represents object of specific type **(defined by the universe of discourse)**
  Examples: x, y
  (universe of discourse can be people, students, cars)

- **Functions** applied to one or more terms
  Examples:. *father-of (John), father-of(father-of(John))*

First order logic. Syntax.

- **Terms do not define propositions (they cannot be evaluated to True or False)**

Sentences in FOL → define propositions:

- **Atomic sentences:**
  - **A predicate symbol** applied to 0 or more terms

    Examples:

    Red(car12),

    Sister(Amy, Jane);

    Manager(father-of(John));
  - t1 = t2 **equivalence** of terms

    Example:

    John = father-of(Peter)

First order logic. Syntax.

Sentences in FOL:

- **Complex sentences:**
- Assume ϕ, ψ are sentences in FOL. Then:
  - $(\phi \wedge \psi)$ $(\phi \vee \psi)$ $(\phi \Rightarrow \psi)$ $(\phi \Leftrightarrow \psi)$ $\neg \psi$

    and
  - $\forall x\ \phi$ $\exists y\ \phi$

  are sentences

Symbols $\exists, \forall$

- stand for the **existential** and the **universal** quantifier

Semantics. Interpretation.

An interpretation ***I*** is defined by a **mapping** to the **domain of discourse D** or **relations on D**

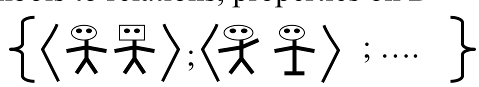
- **domain of discourse:** a set of objects in the world we represent and refer to;

An interpretation *I* maps:

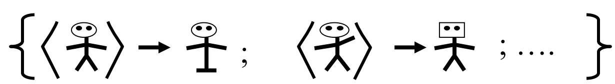
- Constant symbols to objects in D

  I(*John*) = 

- Predicate symbols to relations, properties on D

  I(*brother*) = 

- Function symbols to functional relations on D

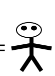
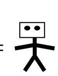
  I(*father-of*) = 

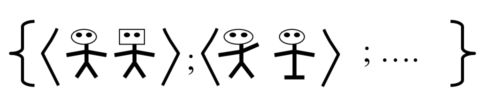
Semantics of sentences.

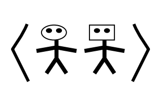
Meaning (evaluation) function:

V : sentence $\times$ interpretation $\rightarrow$ {*True* , *False* }

A **predicate** *predicate(term-1, term-2, term-3, term-n)* is true for the interpretation *I* , iff the objects referred to by *term-1, term-2, term-3, term-n* are in the relation referred to by *predicate*

I(*John*) =  I(*Paul*) = 

I(*brother*) = 

brother(John, Paul) =  in I(*brother*)

V(*brother(John, Paul), I*) = ***True***

Semantics of sentences.

- **Equality** V(*term-1*= *term-2, I*) = ***True***
  Iff I(*term-1*) =I(*term-2*)
- **Boolean expressions: standard**

  E.g. V(*sentence-1* $\vee$ *sentence-2, I*) = ***True***
  Iff V(*sentence-1,I*)= ***True*** or V(*sentence-2,I*)= ***True***

- **Quantifications**

  V($\forall x\ \phi$, *I*) = ***True***
  Iff for all $d \in D$ V(ϕ ,*I[x/d]*)= ***True*** (*I[x/d]*: substitution of *x* with *d*)

  V($\exists x\ \phi$, *I*) = ***True***
  Iff there is a $d \in D$, s.t. V(ϕ ,*I[x/d]*)= ***True***

Sentences with quantifiers

- **Universal quantification**

  All Upitt students are smart

- **Assume the universe of discourse of x are Upitt students**

Sentences with quantifiers

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  $\forall x \; smart(x)$

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Sentences with quantifiers

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- **Assume the universe of discourse of x are Upitt students**

  $$\forall x \; smart(x)$$

- **Assume the universe of discourse of x are students**

  $$\forall x \; at(x, Upitt) \Rightarrow smart(x)$$

Sentences with quantifiers

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- **Assume the universe of discourse of x are people**

Sentences with quantifiers

- **Universal quantification**

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$$\forall x \; smart(x)$$

- **Assume the universe of discourse of x are students**

$$\forall x \; at(x, Upitt) \Rightarrow smart(x)$$

- **Assume the universe of discourse of x are people**

$$\forall x \; student(x) \wedge at(x, Upitt) \Rightarrow smart(x)$$

Sentences with quantifiers

- **Universal quantification**

All Upitt students are smart

- **Assume the universe of discourse of x are Upitt students**

$$\forall x \; smart(x)$$

- **Assume the universe of discourse of x are students**

$$\forall x \; at(x, Upitt) \Rightarrow smart(x)$$

- **Assume the universe of discourse of x are people**

$$\forall x \; student(x) \wedge at(x, Upitt) \Rightarrow smart(x)$$

Typically the universal quantifier connects with an implication

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- **Assume the universe of discourse of x are CMU affiliates**

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- **Assume the universe of discourse of x are CMU affiliates**

$$\exists x \ at(x, CMU) \quad \wedge \ smart(x)$$

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- **Assume the universe of discourse of x are CMU affiliates**

$\exists x \; at(x,CMU) \quad \wedge \; smart(x)$

- **Assume the universe of discourse of x are people**

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- **Assume the universe of discourse of x are CMU affiliates**

$\exists x \; smart(x)$

- **Assume the universe of discourse of x are people**

$\exists x \; at(x,CMU) \quad \wedge \; smart(x)$

Sentences with quantifiers

- **Existential quantification**

Someone at CMU is smart

- **Assume the universe of discourse of x are CMU affiliates**

$\exists x \; at(x, CMU) \quad \wedge smart(x)$

- **Assume the universe of discourse of x are people**

$\exists x \; at(x, CMU) \quad \wedge smart(x)$

Typically the existential quantifier connects with a conjunction

Order of quantifiers

- **Order of quantifiers of the same type does not matter**

For all x and y, if x is a parent of y then y is a child of x

$$\forall x, y \;\; parent \;\; (x, y) \Rightarrow \; child \;\; (y, \mathrm{x})$$

$$\forall y, x \;\; parent \;\; (x, y) \Rightarrow \; child \;\; (y, \mathrm{x})$$

- **Order of different quantifiers changes the meaning**

$$\forall x \exists y \; loves \;\; (x, y)$$

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- **Order of quantifiers of the same type does not matter**

  For all x and y, if x is a parent of y then y is a child of x

$$\forall x, y \ \ parent \ (x, y) \Rightarrow child \ (y, x)$$
$$\forall y, x \ \ parent \ (x, y) \Rightarrow child \ (y, x)$$

- **Order of different quantifiers changes the meaning**

$$\forall x \exists y \ \ loves \ (x, y)$$

Everybody loves somebody

$$\exists y \forall x \ \ loves \ (x, y)$$

Order of quantifiers

- **Order of quantifiers of the same type does not matter**

  For all x and y, if x is a parent of y then y is a child of x

$$\forall x, y \ \ parent \ (x, y) \Rightarrow child \ (y, x)$$
$$\forall y, x \ \ parent \ (x, y) \Rightarrow child \ (y, x)$$

- **Order of different quantifiers changes the meaning**

$$\forall x \exists y \ \ loves \ (x, y)$$

Everybody loves somebody

$$\exists y \forall x \ \ loves \ (x, y)$$

There is someone who is loved by everyone

Connections between quantifiers

Everyone likes ice cream

?

Connections between quantifiers

Everyone likes ice cream

$$\forall x \; likes(x, IceCream)$$

Connections between quantifiers

Everyone likes ice cream

$\forall x \; likes(x, IceCream)$

Is it possible to convey the same meaning using an existential quantifier ?

Connections between quantifiers

Everyone likes ice cream

$\forall x \; likes(x, IceCream)$

Is it possible to convey the same meaning using an existential quantifier ?

There is no one who does not like ice cream

$\neg \exists x \; \neg likes(x, IceCream)$

A universal quantifier in the sentence can be expressed using an existential quantifier !!!

Connections between quantifiers

Someone likes ice cream

?

Connections between quantifiers

Someone likes ice cream

$$\exists x \; likes(x, IceCream)$$

Is it possible to convey the same meaning using a universal quantifier ?

Connections between quantifiers

Someone likes ice cream

$$\exists x \; likes(x, IceCream)$$

Is it possible to convey the same meaning using a universal quantifier ?

Not everyone does not like ice cream

$$\neg \forall x \; \neg likes(x, IceCream)$$

An existential quantifier in the sentence can be expressed using a universal quantifier !!!

Representing knowledge in FOL

Example:

Kinship domain

- **Objects:** people

  $John, Mary, Jane, \ldots$
- **Properties:** gender

  $Male(x), Female(x)$
- **Relations:** parenthood, brotherhood, marriage

  $Parent(x, y), Brother(x, y), Spouse(x, y)$
- **Functions:** mother-of (one for each person x)

  $MotherOf(x)$

Kinship domain in FOL

Relations between predicates and functions: write down what we know about them; how relate to each other.

- Male and female are disjoint categories

$$\forall x \; Male(x) \Leftrightarrow \neg Female(x)$$

- Parent and child relations are inverse

$$\forall x, y \; Parent(x, y) \Leftrightarrow Child(y, x)$$

- A grandparent is a parent of parent

$$\forall g, c \; Grandparent(g, c) \Leftrightarrow \exists p \; Parent(g, p) \wedge Parent(p, c)$$

- A sibling is another child of one's parents

$$\forall x, y \; Sibling(x, y) \Leftrightarrow (x \neq y) \wedge \exists p \; Parent(p, x) \wedge Parent(p, y)$$

- And so on ….

Inference in First order logic

Logical inference in FOL

Logical inference problem:

- Given a knowledge base KB (a set of sentences) and a sentence α , does the KB semantically entail α?

$$KB \models \alpha \quad ?$$

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Logical inference problem in the first-order logic is undecidable !!!. No procedure that can decide the entailment <u>for all possible input sentences</u> in a finite number of steps.

Logical inference problem in the Propositional logic

Computational procedures that answer:

$$KB \models \alpha \quad ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**

Inference in FOL: Truth table

- Is the Truth-table approach a viable approach for the FOL?

?

Inference in FOL: Truth table approach

- Is the Truth-table approach a viable approach for the FOL?

?

- NO!
- Why?
- It would require us to enumerate and list all possible interpretations I
- I = (assignments of symbols to objects, predicates to relations and functions to relational mappings)
- Simply there are too many interpretations

Inference in FOL: Inference rules

- **Is the Inference rule approach a viable approach for the FOL?**

?

Inference in FOL: Inference rules

- **Is the Inference rule approach a viable approach for the FOL?**

?

- **Yes.**
- **The inference rules represent sound inference patterns one can apply to sentences in the KB**
- **What is derived follows from the KB**
- **Caveat: we need to add rules for handling quantifiers**

Inference rules

- **Inference rules from the propositional logic:**
  - Modus ponens

$$\frac{A \Rightarrow B, \quad A}{B}$$

  - Resolution

$$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

  - and others: And-introduction, And-elimination, Or-introduction, Negation elimination
- **Additional inference rules** are needed for sentences with quantifiers and variables
  - Must involve variable substitutions

Sentences with variables

First-order logic sentences can include variables.

- **Variable** is:
  - **Bound** – if it is in the scope of some quantifier

$$\forall x \;\; P(x)$$

  - **Free** – if it is not bound.

$$\exists x \;\; P(y) \wedge Q(x) \qquad y \text{ is free}$$

- **Sentence** (formula) is:
  - **Closed** – if it has no free variables

$$\forall y \exists x \; P(y) \Rightarrow Q(x)$$

  - **Open** – if it is not closed
  - **Ground** – if it does not have any variables

$$Likes(John, Jane)$$

Variable substitutions

- Variables in the sentences can be substituted with terms.
  (terms = constants, variables, functions)
- **Substitution:**
  - Is represented by a mapping from variables to terms

    $\{x_1 / t_1, x_2 / t_2, \ldots\}$

  - Application of the substitution to sentences

$$SUBST(\{x / Sam, y / Pam\}, Likes(x, y)) = Likes(Sam, Pam)$$

$$SUBST(\{x / z, y / fatherof(John)\}, Likes(x, y)) = Likes(z, fatherof(John))$$

Inference rules for quantifiers

- **Universal elimination**

$$\frac{\forall x\ \phi(x)}{\phi(a)} \qquad a \text{ - is a constant symbol}$$

  - substitutes a variable with a constant symbol

$\forall x\ Likes(x, IceCream) \qquad Likes(Ben, IceCream)$

- **Existential elimination.**

$$\frac{\exists x\ \phi(x)}{\phi(a)}$$

  - Substitutes a variable with a constant symbol that does not appear elsewhere in the KB

$\exists x\ Kill(x, Victim) \qquad Kill(Murderer, Victim)$