

CS 2710 Foundations of AI

Lecture 20-a

Utility theory

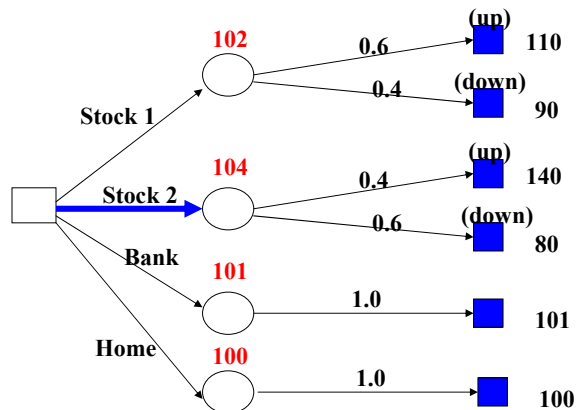
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Selection based on expected values

- **Until now:** The optimal action choice was the option that maximized the expected monetary value.
- **But is the expected monetary value always the quantity we want to optimize?**



Selection based on expected values

- **Is the expected monetary value always the quantity we want to optimize?**
- **Answer:** Yes, but only if we are risk-neutral.
- But what if **we do not like the risk (we are risk-averse)?**
- In that case we may want to get the premium for undertaking the risk (of losing the money)
- **Example:**
 - we may prefer to get \$101 for sure against \$102 in expectation but with the risk of losing the money
- **Problem:** How to model decisions and account for the risk?
- **Solution:** use **utility function, and utility theory**

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Utility function

- **Utility function (denoted U)**
 - Quantifies how we “value” outcomes, i.e., it reflects our preferences
 - Can be also applied to “value” outcomes other than money and gains (e.g. utility of a patient being healthy, or ill)
- **Decision making:**
 - uses expected utilities (denoted EU)

$$EU(X) = \sum_{x \in \Omega_x} P(X = x)U(X = x)$$

$U(X = x)$ the utility of outcome x

Important !!!

- Under some conditions on preferences **we can always design the utility function that fits our preferences**

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Utility theory

- Defines axioms on preferences that involve uncertainty and ways to manipulate them.
- Uncertainty is modeled through **lotteries**
 - **Lottery:**
$$[p : A; (1 - p) : C]$$
 - Outcome A with probability p
 - Outcome C with probability (1-p)
- The following six constraints are known as the axioms of utility theory. The axioms are the most obvious semantic constraints on preferences with lotteries.
- **Notation:**
 - \succ - preferable
 - \sim - indifferent (equally preferable)

Axioms of the utility theory

- **Orderability:** Given any two states, the a rational agent prefers one of them, else the two as equally preferable.
$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$
- **Transitivity:** Given any three states, if an agent prefers A to B and prefers B to C , agent must prefer A to C .
$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$
- **Continuity:** If some state B is between A and C in preference, then there is a p for which the rational agent will be indifferent between state B and the lottery in which A comes with probability p , C with probability $(1-p)$.

$$(A \succ B \succ C) \Rightarrow \exists p [p : A; (1 - p) : C] \sim B$$

Axioms of the utility theory

- **Substitutability:** If an agent is indifferent between two lotteries, A and B , then there is a more complex lottery in which A can be substituted with B .

$$(A \sim B) \Rightarrow [p : A; (1 - p) : C] \sim [p : B; (1 - p) : C]$$

- **Monotonicity:** If an agent prefers A to B , then the agent must prefer the lottery in which A occurs with a higher probability

$$(A \succ B) \Rightarrow (p > q \Leftrightarrow [p : A; (1 - p) : B] \succ [q : A; (1 - q) : B])$$

- **Decomposability:** Compound lotteries can be reduced to simpler lotteries using the laws of probability.

$$[p : A; (1 - p) : [q : B; (1 - q) : C]] \Rightarrow [p : A; (1 - p)q : B; (1 - p)(1 - q) : C]$$

Utility theory

If the agent obeys the axioms of the utility theory, then

1. there exists a real valued function U such that:

$$U(A) > U(B) \Leftrightarrow A \succ B$$

$$U(A) = U(B) \Leftrightarrow A \sim B$$

2. The utility of the lottery is the expected utility, that is the sum of utilities of outcomes weighted by their probability

$$U[p : A; (1 - p) : B] = pU(A) + (1 - p)U(B)$$

3. Rational agent makes the decisions in the presence of uncertainty by maximizing its expected utility

Utility functions

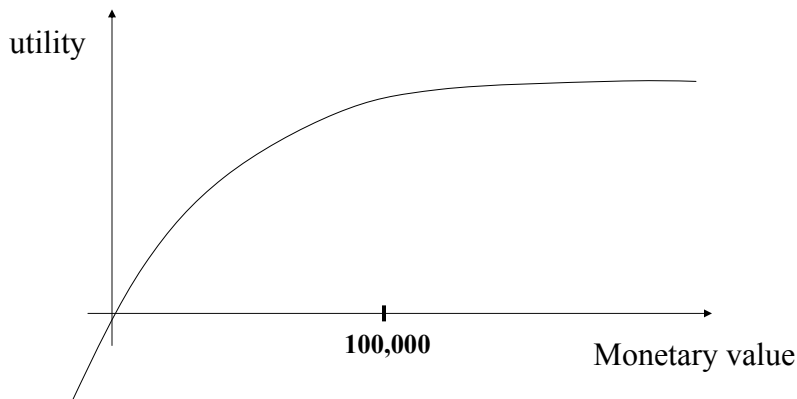
We can design a utility function that fits our preferences if they satisfy the axioms of utility theory.

- **But how to design the utility function for monetary values so that they incorporate the risk?**
- **What is the relation between utility function and monetary values?**
- Assume we loose or gain \$1000.
 - Typically this difference is more significant for lower values (around \$100 -1000) than for higher values (~ \$1,000,000)
- What is the relation between utilities and monetary value for a typical person?

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Utility functions

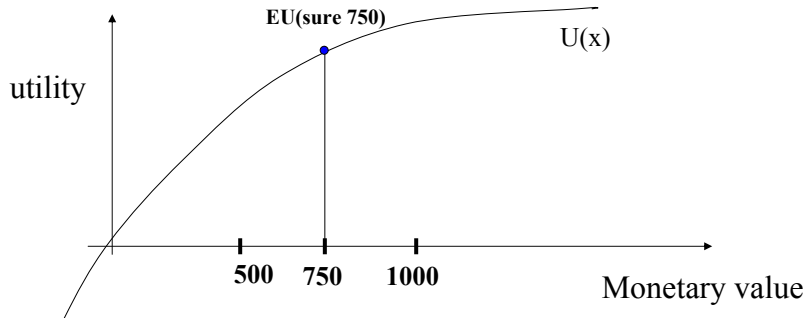
- What is the relation between utilities and monetary value for a typical person?
- Concave function that flattens at higher monetary values



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Utility functions

- Expected utility of a sure outcome of 750 is 750

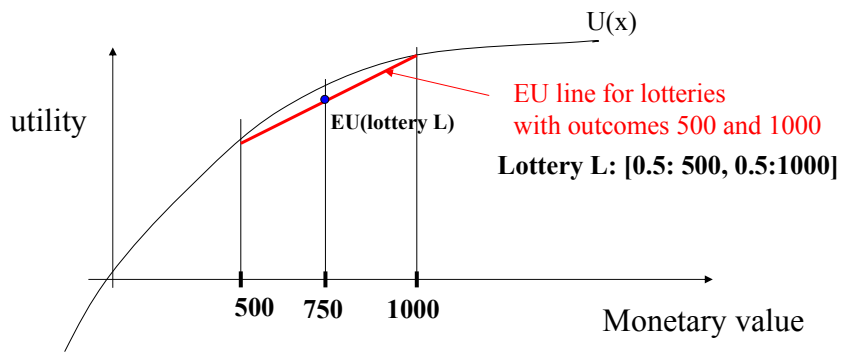


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Utility functions

Assume a lottery L [0.5: 500, 0.5:1000]

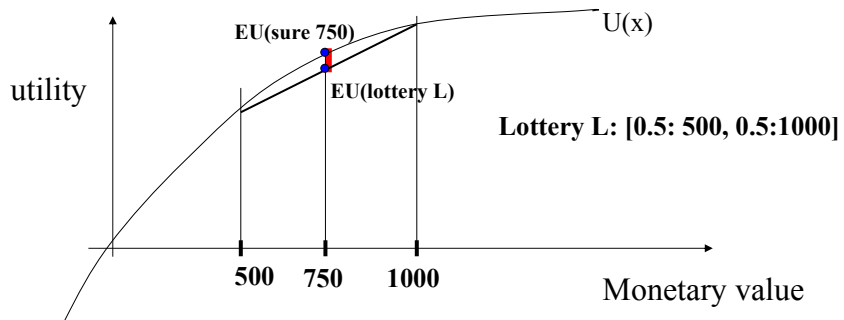
- Expected value of the lottery = 750
- Expected utility of the lottery $EU(L)$ is different:
 - $EU(L) = 0.5U(500) + 0.5*U(1000)$



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Utility functions

- Expected utility of the lottery $EU(\text{lottery } L) < EU(\text{sure } 750)$



- Risk aversion – a bonus is required for undertaking the risk