Section 2.2
Set Operations

Propositional calculus and set theory are both instances of an algebraic system called a

Boolean Algebra.

The operators in set theory are defined in terms of the corresponding operator in propositional calculus

As always there must be a universe U. All sets are assumed to be subsets of U

Definition: Two sets A and B are *equal*, denoted $A = B$, iff

$$x[x \quad A \leftrightarrow x \quad B].$$

Note: By a previous logical equivalence we have

$$\text{A = B iff } \quad x[(x \quad A \rightarrow x \quad B) \quad (x \quad B \rightarrow x \quad A)]$$

or

$$\text{A = B iff } A \quad B \text{ and } B \quad A$$

Definitions:

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- The *union* of A and B, denoted A   B, is the set

{x | x A x B}

- The *intersection* of A and B, denoted A ∩ B, is the set

{x | x A x B}

Note: If the intersection is void, A and B are said to be *disjoint.*

- The *complement* of A, denoted $\overline{A}$, is the set

{x | ¬ (x A)}

Note: Alternative notation is A^c, and {x|x A}.

- The *difference* of A and B, or the *complement* of B *relative to* A, denoted A - B, is the set

$$A \cap \overline{B}$$

Note: The (absolute) complement of A is U - A.

- The *symmetric difference* of A and B, denoted *A*   *B*, is the set

$$(A - B) \quad (B - A)$$

Examples: U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

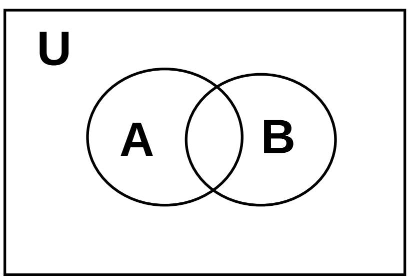
A= {1, 2, 3, 4, 5}, B = {4, 5, 6, 7, 8}. Then

- $A \quad B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A \cap B = \{4, 5\}$
- $\overline{A} = \{0, 6, 7, 8, 9, 10\}$
- $\overline{B} = \{0, 1, 2, 3, 9, 10\}$
- $A - B = \{1, 2, 3\}$
- $B - A = \{6, 7, 8\}$
- $A \quad B = \{1, 2, 3, 6, 7, 8\}$

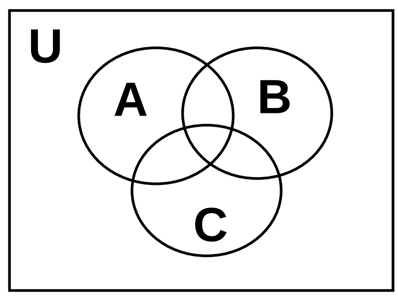
Venn Diagrams

A useful geometric visualization tool (for 3 or less sets)

- The Universe U is the rectangular box
- Each set is represented by a circle and its interior
- All possible combinations of the sets must be represented



For 2 sets



For 3 sets

Shade the appropriate region to represent the given set operation.

Set Identities

Set identities correspond to the logical equivalences.

Example:

The complement of the union is the intersection of the complements:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Proof: To show:

$$\forall x[x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}]$$

To show two sets are equal we show for all x that x is a member of one set if and only if it is a member of the other.

We now apply an important *rule of inference* (defined later) called

Universal Instantiation

In a proof we can eliminate the universal quantifier which binds a variable if we do not assume anything about the variable other than it is an arbitrary member of the Universe. We can then treat the resulting predicate as a proposition.

We say

'Let x be arbitrary.'

<u>Then</u> we can treat the predicates as propositions:

Assertion	Reason
$x \quad \overline{A \quad B} \Leftrightarrow x \quad [A \quad B]$	Def. of complement
$x \quad A \quad B \Leftrightarrow \neg[x \quad A \quad B]$	Def. of
$\Leftrightarrow \neg[x \quad A \quad x \quad B]$	Def. of union
$\Leftrightarrow \neg x \quad A \quad \neg x \quad B$	DeMorgan's Laws
$\Leftrightarrow x \quad A \quad x \quad B$	Def. of
$\Leftrightarrow x \quad \overline{A} \quad x \quad \overline{B}$	Def. of complement
$\Leftrightarrow x \quad \overline{A} \cap \overline{B}$	Def. of intersection

Hence

$$x \quad \overline{A \quad B} \leftrightarrow x \quad \overline{A} \cap \overline{B}$$

is a tautology.

Since

- x was arbitrary
- we have used only logically equivalent assertions and definitions

we can apply another rule of inference called

Universal Generalization

We can apply a universal quantifier to bind a variable if we have shown the predicate to be true for all values of the variable in the Universe.

and claim the assertion is true for all x, i.e.,

$$x[x \quad \overline{A \quad B} \leftrightarrow x \quad \overline{A} \cap \overline{B}]$$

Q. E. D. (an abbreviation for the Latin phrase “Quod Erat Demonstrandum” - “which was to be demonstrated” used to signal the end of a proof)

Note: As an alternative which might be easier in some cases, use the identity

$$A = B \Leftrightarrow [A \quad B \text{ and } B \quad A]$$

Example:

Show $A \cap (B - A) =$

The void set is a subset of every set. Hence,

$$A \cap (B - A)$$

Therefore, it suffices to show

$$A \cap (B - A)$$

or

$$x[x \quad A \cap (B - A) \rightarrow x \qquad]$$

So as before we say 'let x be arbitrary'.

Show

$$\mathrm{x} \quad \mathrm{A} \cap (\mathrm{B} - \mathrm{A}) \rightarrow \mathrm{x}$$

is a tautology.

But the consequent is always false.

Therefore, the antecedent better always be false also.

Apply the definitions:

Assertion	Reason
$\mathrm{x} \quad \mathrm{A} \cap (\mathrm{B} - \mathrm{A}) \Leftrightarrow \mathrm{x} \quad \mathrm{A} \quad \mathrm{x} \quad (\mathrm{B} - \mathrm{A})$	Def. of $\cap$
$\Leftrightarrow \mathrm{x} \quad \mathrm{A} \quad (\mathrm{x} \quad \mathrm{B} \quad \mathrm{x} \quad \mathrm{A})$	Def. of -
$\Leftrightarrow (x \quad A \quad x \quad A) \quad x \quad B$	Props of 'and'
$\Leftrightarrow 0 \quad x \quad B$	Table 6
$\Leftrightarrow 0$	Domination

Hence, because $P \quad \neg P$ is always false, the implication is a tautology.

The result follows by Universal Generalization.

Q. E. D.

Union and Intersection of Indexed Collections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

Union and intersection are associative (because 'and' and 'or' are) we have:

$$\bigcup_{i=1}^{n} A_i = A_1 \quad A_2 \quad \ldots \quad A_n$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Examples:

Let

$$A_i = [i, \infty), 1 \leq i < \infty$$

$$\bigcup_{i=1}^{n} A_i = [1, \infty)$$

$$\bigcap_{i=1}^{n} A_i = [n, \infty)$$
