# Today's Topics

Introduction to set theory

- What is a set?
- Set notation
- Basic set operations

# What is a set?

***Definition***: A set is an unordered collection of objects

***Examples:***

**Sets can contain items of mixed types**

- A = {1, 2, 3, 4}
- B = {cow, pig, turkey}
- C = {motorcycle, 3.14159, Socrates}
- E = {{1, 2, 3}, {6, 7, 8}, {23, 42}}

**Sets can contain other sets**

***Informally:*** Sets are really just a precise way of grouping a "bunch of stuff"

## A set is made up of elements

***Definition:*** The objects making up a set are called elements of that set.

***Examples:***

- 3 is an element of {1, 2, 3}
- Bob is an element of {Alice, Bob, Charlie, Daniel}

We can express the above examples in a more precise manner as follows:

- $3 \in \{1, 2, 3\}$
- Bob ∈ {Alice, Bob, Charlie, Daniel}

***Question:*** Is $5 \in \{1, 2, 3, \{4, 5\}\}$?

## There are many different ways to describe a set

***Explicit enumeration:***

- $A = \{1, 2, 3, 4\}$

***Using ellipses if the general pattern is obvious:***

- $E = \{2, 4, 6, \ldots, 98\}$

***Set builder notation:***

- $M = \{y \mid y = 3k \text{ for some integer } k\}$

**The set M contains…**
**… all elements y…**
**… such that…**
**… y = 3k for some integer k**

## There are a number of sets that are so important to mathematics that they get their own symbol

**N** = {0, 1, 2, 3, …}

**Z** = {…, -2, -1, 0, 1, 2, …}

**Z**$^+$ = {1, 2, …}

**Q** = {p/q | p,q ∈ **Z**, q≠0}

**R**

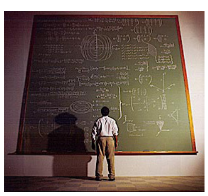
∅ = {}

Note: This notation differs from book to book

- Some authors do not include zero in the natural numbers

Be careful when reading other books or researching on the Web, as things may be slightly different!

## You've actually been using sets implicitly all along!



Mathematics

F(x,y) ≡ x and y are friends
Domain: "All people"

∀x ∃y F(x,y)

Domains of propositional functions

```
Function min(int x, int y) : int
   if x < y then
      return x
   else
      return y
   endif
end function
```

Programming language data types

## Set equality

***Definition:*** Two sets are equal if and only if they contain exactly the same elements.

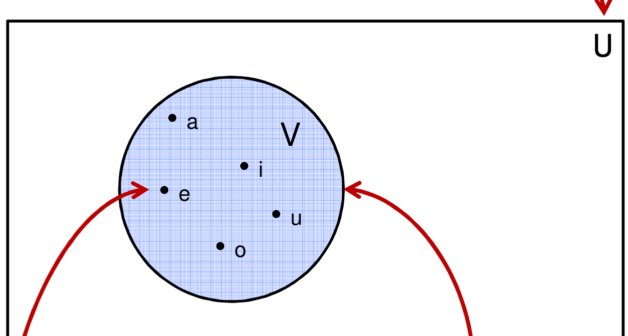
***Mathematically:*** $A = B$ iff $\forall x\ (x \in A \leftrightarrow x \in B)$

***Example:*** Are the following sets equal?

- {1, 2, 3, 4} and {1, 2, 3, 4}
- {1, 2, 3, 4} and {4, 1, 3, 2}
- {a, b, c, d, e} and {a, a, c, b, e, d}
- {a, e, i, o} and {a, e, i, o, u}

## We can use Venn diagrams to graphically represent sets

**U is the "universe" of all elements**



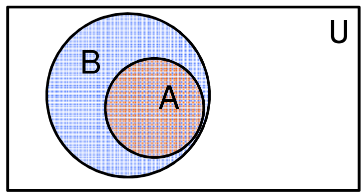

**The set V of all vowels is contained with in the universe of "all letters"**

**Sometimes, we add points for the elements of a set**

## Sets can be contained within one another

***Definition:*** Some set A is a subset of another set B iff every element of A is contained in the set B. We denote this fact as $A \subseteq B$, and call B a superset of A.

***Graphically:***




***Mathematically:***

***Definition:*** We say that A is a proper subset of B iff $A \subseteq B$, but $A \neq B$. We denote this by $A \subset B$. More precisely:

$$A \subset B \text{ iff } \forall x\,(x \in A \rightarrow x \in B) \wedge \exists y\,(y \in B \wedge y \notin A)$$

## Properties of subsets

Property 1: For all sets S, we have that $\emptyset \subseteq S$

***Proof:*** The set $\emptyset$ contains no elements. So, trivially, every element of the set $\emptyset$ is contained in any other set S. ❑

Property 2: For any set S, $S \subseteq S$.

Property 3: If $S_1 = S_2$, then $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$.

## Group work!

**Problem 1:** Come up with two ways to represent each of the following sets:

- The even integers
- Negative numbers between -1 and -10, inclusive
- The positive integers

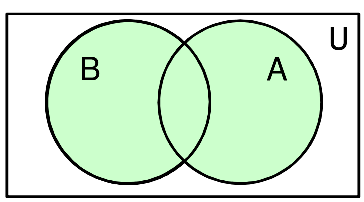
**Problem 2:** Are the sets {a, b, c} and {c, c, a, b, a, b} equal? Why or why not?

**Problem 3:** Draw a Venn diagram representing the sets {1, 2, 3} and {3, 4, 5}.

## We can create a new set by combining two or more existing sets

***Definition:*** The union of two sets A and B contains every element that is either in A or in B. We denote the union of the sets A and B as $A \cup B$.

***Graphically:***



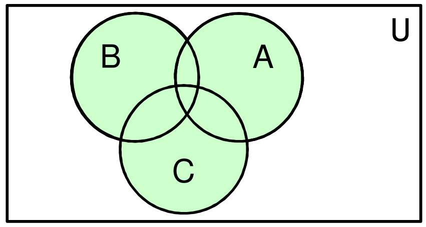

***Mathematically:***

***Example:*** $\{1, 2, 3\} \cup \{6, 7, 8\} = \{1, 2, 3, 6, 7, 8\}$

## We can take the union of any number of sets

***Example:*** A ∪ B ∪ C

***Graphically:***




In general, we can express the union $S_1 \cup S_2 \cup \ldots \cup S_n$ using the following notation:

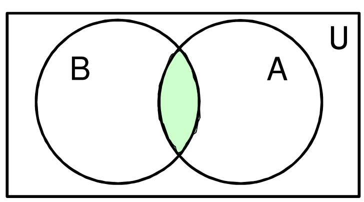
$$\bigcup_{i=1}^{n} S_n$$

**This is just like summation notation!**

## Sometimes we're interested in the elements that are in more than one set

***Definition:*** The intersection of two sets A and B contains every element that is in A and also in B. We denote the intersection of the sets A and B as A ∩ B.

***Graphically:***




***Mathematically:***

***Examples:***

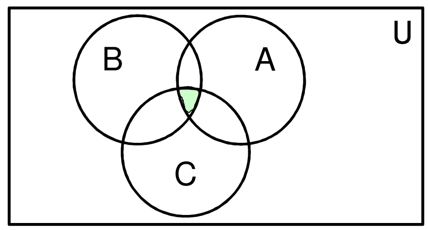
- {1, 2, 3, 7, 8} ∩ {6, 7, 8} = {7, 8}
- {1, 2, 3} ∩ {6, 7, 8} = ∅

**We say that two sets A and B are <u>disjoint</u> if A ∩ B = ∅**

## We can take the intersection of any number of sets

***Example:*** A ∩ B ∩ C

***Graphically:***



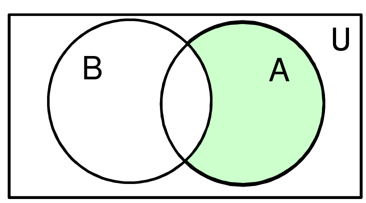

As with the union operation, we can express the intersection $S_1 \cap S_2 \cap \ldots \cap S_n$ as:

$$\bigcap_{i=1}^{n} S_n$$

## Set differences

***Definition:*** The difference of two sets A and B, denoted by A – B, contains every element that is in A, but not in B.

***Graphically:***




***Mathematically:***

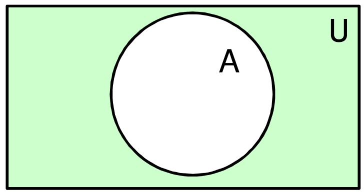
***Example:*** {1, 2, 3, 4, 5} - {4, 5, 6, 7, 8} = {1, 2, 3}

Be careful: Some authors use the notation A \ B to denote the set difference A - B.

## If we have specified a universe U, we can determine the complement of a set

***Definition:*** The complement of a set A, denoted by $\overline{A}$, contains every element that is in U, but not in A.

***Graphically:***




***Mathematically:***

***Examples:*** Assume that U = {1, 2, …, 10}

- $\overline{\{1, 2, 3, 4, 5\}}$ =
- $\overline{\{2, 4, 6, 8, 10\}}$ =

## Cardinality is the measure of a set's size

***Definition:*** Let S be a set. If there are exactly n elements in S, where n is a nonnegative integer, then S is a finite set whose cardinality is n. The cardinality of S is denoted by |S|.

***Example:*** If S = {a, e, i, o, u}, then |S| =

***Useful facts:*** If A and B are finite sets, then

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A - B| = |A| - |A \cap B|$

Aside: We'll talk about the cardinality of infinite sets later in the course.

# Power set

***Definition:*** Given a set S, it's power set is the set containing all subsets of S. We denote the power set of S as P(S).

***Examples:***

- P({1}) = {∅, {1}}
- P({1, 2, 3}) = {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2, 3}, {1, 2, 3}}

Note:

- The set ∅ is always in the power set of any set S
- The set S is always in its own power set
- $|P(S)| = 2^{|S|}$
- Some authors use the notation $2^S$ to represent the power set of S

# How do we represent ordered collections?

***Definition:*** The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, $a_2$ as its second element, …, and $a_n$ as its $n^{th}$ element.

Note: $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ iff $a_i = b_i$ for i = 1, …, n.

Special case: Ordered pairs of the form ($x \in \mathbf{Z}$, $y \in \mathbf{Z}$) are the basis of the Cartesian plane!

- (a, b) = (c, d) iff a = c and b = d
- (a, b) = (b, a) iff a = b

***How can we construct and describe ordered n-tuples?***

## We use the Cartesian product operator to construct ordered n-tuples

***Definition:*** If A and B are sets, the Cartesian product of A and B, which is denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

***Mathematically:***

***Examples:*** Let A = {1, 2} and B = {y, z}

- What is $A \times B$?
- $B \times A$?
- Are $A \times B$ and $B \times A$ equivalent?

## Cartesian products can be made from more than two sets

***Example:*** Let

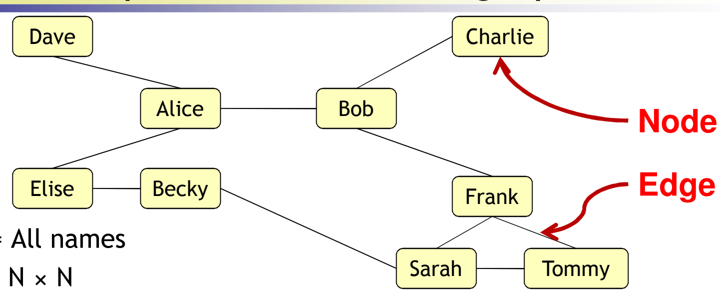
- $S = \{x \mid x \text{ is enrolled in CS 441}\}$
- $G = \{x \mid x \in \mathbf{R} \wedge 0 \leq x \leq 100\}$
- Y = {freshman, sophomore, junior, senior}

The set $S \times Y \times G$ consists of all possible (CS441 student, year, grade) combinations.

Note: My grades database is a subset of $S \times Y \times G$ that defines a relation between students in the class, their year at Pitt, and their grade!

**We will study the properties of relations towards the end of this course.**

## Sets and Cartesian products can be used to represent trees and graphs




***Let:***

- N = All names
- F = N × N

A social network can be represented as a graph (V, E) ∈ N × F in which the set V denotes the people in the network and the set E denotes the set of "friendship" links.

In the above network:

- V = {Alice, Bob, …, Tommy} ⊆ N
- E = {(Alice, Bob), (Alice, Dave), …, (Sarah, Tommy)} ⊆ N × N

## Set notation allows us to make quantified statements more precise

We can use set notation to make the domain of a quantified statement explicit.

***Example:*** $\forall x \in \mathbf{R}\ (x^2 \geq 0)$

- The square of any real number is a least zero

***Example:*** $\forall n \in \mathbf{Z}\ \exists j,k \in \mathbf{Z}\ [(3n+2 = 2j+1) \rightarrow (n = 2k+1)]$

- If n is an integer and 3n + 2 is odd, then n is odd.

Note: This notation is far less ambiguous than simply stating the domains of propositional functions. In the remainder of the course, we will use this notation whenever possible.

## Truth sets describe when a predicate is true

***Definition:*** Given a predicate P and its corresponding domain D the truth set of P enumerates all elements in D that make the predicate P true.

***Examples:*** What are the truth sets of the following predicates, given that their domain is the set **Z**?

- $P(x) \equiv |x| = 1$
- $Q(x) \equiv x^2 > 0$
- $R(x) \equiv x^5 = 1049$

Note:

- $\forall x$ P(x) is true iff the truth set of P is the entire domain D
- $\exists x$ P(x) is true iff the truth set of P is non-empty

## How do computers represent and manipulate finite sets?

Observation: Representing sets as unordered collections of elements (e.g., arrays of Java Object data types) is very inefficient.

As a result, sets are usually represented using either hash maps or bitmaps.

**You'll learn about these in a data structures class, so today we'll focus on bitmap representations.**

This is probably best explained through an example…

## Playing with the set S={x | x∈**N**, x<10}

To represent a set as a bitmap, we must first agree on an ordering for the set. In the case of S, let's use the natural ordering of the numbers.

Now, any subset of S can be represented using |S|=10 bits. For example:

- {1, 3, 5, 7, 9} = 0101 0101 01
- {1, 1, 1, 4, 5} = 0100 1100 00

What subsets of S do the following bitmaps represent?

- 0101 1010 11
- 1111 0000 10

## Set operations can be carried out very efficiently as bitwise operations

***Example:*** {1, 3, 7} ∪ {2, 3, 8}

0101 0001 00
∨ 0011 0000 10
10

= {1, 2, 3, 7, 8}

***Example:*** {1, 3, 7} ∩ {2, 3, 8}

0101 0001 00
∧ 0011 0000 10
0001 0000 00

= {3}

Note: These operations are much faster than searching through

## Set operations can be carried out very efficiently as bitwise operations

***Example:*** $\overline{\{1, 3, 7\}}$

$\neg$0101 0001 00

1010 1110 11

= {0, 2, 4, 5, 6, 8, 9}

Since the set difference A – B can be written as A ∩ (A ∩ B), we can calculate it as A ∧ ¬(A ∧ B).

**Although set difference is more complicated than the basic operations, it is still much faster to calculate set differences using a bitmap approach as opposed to an unordered search.**

## Group work!

**Problem 1:** Let A = {1, 2, 3, 4}, B = {3, 5, 7, 9}, and C = {7, 8, 9, 10}. Calculate the following:

- A ∩ B
- A ∪ B ∪ C
- B ∩ C
- A ∩ B ∩ C

**Problem 2:** Come up with a bitmap representation of the sets A = {a, c, d, f} and B = {a, b, c}. Use this to calculate the following:

- A ∪ B
- A ∩ B

# Final thoughts

- Sets are one of the most basic data structures used in computer science
- Today, we looked at:
  - How to define sets
  - Basic set operations
  - How computers represent sets
- Next time:
  - Set identities (Section 2.2)
  - Functions (Section 2.3)