Review: Potential stumbling blocks…

Whether the negation sign is on the inside or the outside of a quantified statement makes a big difference!

Example: Let T(x) ≡ "x is tall". Consider the following:

- $\neg \forall x\ T(x)$
  - ↸ "It is not the case that all people are tall."
- $\forall x\ \neg T(x)$
  - ↸ "For all people x, it is not the case that x is tall."

Note: $\neg \forall x\ T(x) = \exists x\ \neg T(x) \neq \forall x\ \neg T(x)$

Recall: When we push negation into a quantifier, DeMorgan's law says that we need to switch the quantifier!

Review: Potential stumbling blocks…

Let: C(x) ≡ "x is enrolled in CS441"

S(x) ≡ "x is smart."

Question: The following two statements look the same, what's the difference?

- $\exists x\ [C(x) \wedge S(x)]$ ← **There is a smart student in CS441.**
- $\exists x\ [C(x) \rightarrow S(x)]$ ← **There exists a student x such that if x is in CS441, then x is smart.**

Subtle note: The second statement is true if there exists even one smart person on Earth, because F→T.

Review: Translation

- Suppose:
  - Variables x,y denote people
  - L(x,y) denotes "x loves y"

- Translate:
  - Everybody loves Raymond
  - Everybody loves somebody.
  - There is somebody whom everybody loves.
  - There is somebody who Raymond doesn't love.
  - There is somebody whom no one loves.
  - Everybody loves himself.

Review: Evaluate

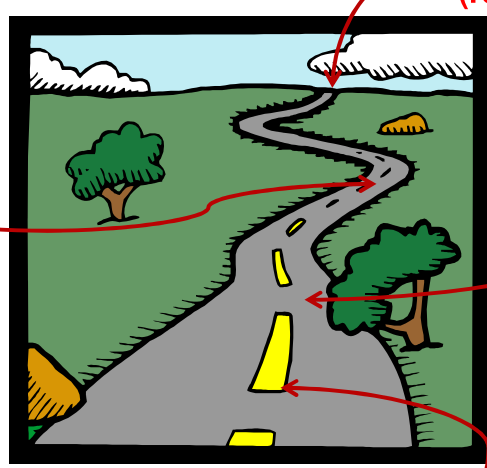
- Domain of discourse = positive integers.
- Let Q(x,y) denote x*x = 2*y
- Let T(x,y) denote x^2 = x*y

- Q(4,8)
- $\exists x\ Q(x,50)$
- $\forall x\ Q(x,x)$
- $\exists x\ \exists y\ Q(x,y)$
- $\exists x\ \forall y\ Q(x,y)$
- $\forall x\ \exists y\ Q(x,y)$
- $\forall x\ \forall y\ Q(x,y)$

Today's topic

- Rules of inference

What have we learned? Where are we going?



Writing valid proofs is a subtle art

Step 1: Discover and formalize the property that you wish to prove

This is called "research"

Step 2: Formalize the ground truths (axioms) that you will use to prove this property

Subtle, but not terribly difficult

Step 3: Show that the property in question follows from the truth of your axioms

Oh, the errors you will make ☺

What is science without jargon?

A conjecture is a statement that is thought to be true.

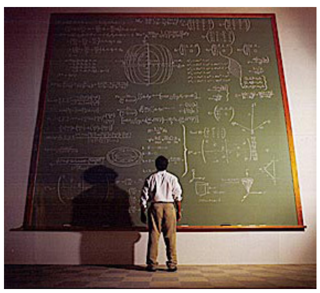
A proof is a valid argument that establishes the truth of a given statement (i.e., a conjecture)

After a proof has been found for a given conjecture, it becomes a theorem

A tale of two proof techniques

In a formal proof, each step of the proof clearly follows from the postulates and axioms assumed in the conjecture.

Statements that are assumed to be true



In an informal proof, one step in the proof may consist of multiple derivations, portions of the proof may be skipped or assumed correct, and axioms may not be explicitly stated.

How can we formalize an argument?

Consider the following argument:

"If you have an account, you can access the network"
"You have an account"
Therefore,
"You can access the network"

This argument *seems* valid, but how can we demonstrate this formally??

Let's analyze the *form* of our argument

p q

"If you have an account, then you can access the network"
"You have an account"
Therefore,
"You can access the network"

This is called a "rule of inference"

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

Rules of inference allow us to make valid arguments

- Many times, we can determine whether an argument is valid by using a truth table, but this is often a cumbersome approach
- Instead, we can apply a sequence of rules of inference to draw valid conclusions from a set of premises

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

Let's analyze the *form* of our argument

"If you have an account, you can access the network"
"You have an account"
Therefore,
"You can access the network"

$p \to q$
p
$\therefore q$

This form is equivalent to the statement
$((p \to q) \land p) \to q$

Since $((p \to q) \land p) \to q$ is a tautology, we know that our argument is valid!

Rules of inference are logically valid ways to draw conclusions when constructing a formal proof

The previous rule is called modus ponens

- Rule of inference: $p \to q$
  p
  $\therefore q$
- Informally: Given an implication $p \to q$, if we know that p is true, then q is also true

But why can we trust modus ponens?

- Tautology: $((p \to q) \land p) \to q$
- Truth table:

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Any time that p→q and p are both true, q is also true!

There are lots of other rules of inference that we can use!

Addition

- *Tautology*: $p \rightarrow (p \vee q)$
- *Rule of inference*:

$$\frac{p}{\therefore p \vee q}$$

- *Example*: "It is raining now, therefore it is raining now or it is snowing now."

Simplification

- *Tautology*: $p \wedge q \rightarrow p$
- *Rule of inference*:
- *Example*: "It is cold outside and it is snowing. Therefore, it is cold outside."

There are lots of other rules of inference that we can use!

Modus tollens

- *Tautology*: $[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$
- *Rule of inference*:
- *Example*: "If I am hungry, then I will eat. I am not eating. Therefore, I am not hungry."

Hypothetical syllogism

- *Tautology*: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
- *Rule of inference*:
- *Example*: "If I eat a big meal, then I feel full. If I feel full, then I am happy. Therefore, if I eat a big meal, then I am happy."

There are lots of other rules of inference that we can use!

Disjunctive syllogism

- *Tautology*: $[\neg p \wedge (p \vee q)] \rightarrow q$
- *Rule of inference*:

- *Example*: "Either the heat is broken, or I have a fever. The heat is not broken, therefore I have a fever."

Conjunction

- *Tautology*: $[(p) \wedge (q)] \rightarrow (p \wedge q)$
- *Rule of inference*:

- *Example*: "Jack is tall. Jack is skinny. Therefore, Jack is tall and skinny."

There are lots of other rules of inference that we can use!

Resolution

- *Tautology*: $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$
- *Rule of inference*:

- *Example*: "If it is not raining, I will ride my bike. If it is raining, I will lift weights. Therefore, I will either ride my bike or lift weights"

Special cases:

1. If $r = q$, we get

2. If $r = F$, we get

We can use rules of inference to build valid arguments

If it is raining, I will stay inside. If am inside, Stephanie will come over. If Stephanie comes over and it is a Saturday, then we will play Scrabble. Today is Saturday. It is raining.

Let:

- r ≡ It is raining
- i ≡ I am inside
- s ≡ Stephanie will come over
- c ≡ we will play Scrabble
- a ≡ it is Saturday

We can use rules of inference to build valid arguments

Let:

- r ≡ It is raining
- i ≡ I am inside
- s ≡ Stephanie will come over
- c ≡ we will play Scrabble
- a ≡ it is Saturday

Hypotheses:

Step:

1.
2.
3.
4.
5.
6.
7.
8.
9.

We also have rules of inference for statements with quantifiers

Universal Instantiation

- Intuition: If we know that P(x) is true for all x, then P(c) is true for a particular c
- Rule of inference:

Universal Generalization

- Intuition: If we can show that P(c) is true for an arbitrary c, then we can conclude that P(x) is true for any x
- Rule of inference:

Note that "arbitrary" does not mean "randomly chosen." It means that we cannot make <u>any</u> assumptions about c other than the fact that it comes from the appropriate domain.

We also have rules of inference for statements with quantifiers

Existential Instantiation

- Intuition: If we know that $\exists$ P(x) is true, then we know that P(c) is true for some c
- Rule of inference:

Again, we cannot make assumptions about c other than the fact that it exists and is from the appropriate domain.

Existential Generalization

- Intuition: If we can show that P(c) is true for a particular c, then we can conclude that $\exists$ P(x) is true
- Rule of inference:

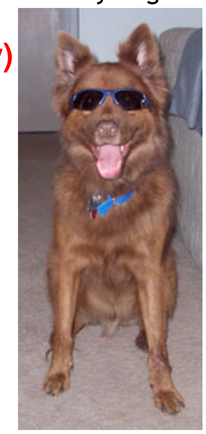
Hungry dogs redux

Given: All of my dogs like peanut butter

M(x) P(x)

Given: Kody is one of my dogs

M(Kody)



1.
2.
3.
4.

Reasoning about our class

Show that the premises "A student in this class has not read the book" and "everyone in this class turned in HW1" imply the conclusion "Someone who turned in HW1 has not read the book."

Let:

-
-
-

Premises:

-
-

Reasoning about our class

Let:

- C(x) ≡ x is in this class
- B(x) ≡ x has read the book
- T(x) ≡ x turned in HW1

Premises:

- $\exists x\, [C(x) \land \neg B(x)]$
- $\forall x\, [C(x) \rightarrow T(x)]$

Steps:

1.
2.
3.
4.
5.
6.
7.
8.
9.

Group work!

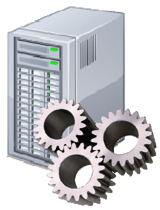
Problem 1: Which rules of inference were used to make the following arguments?

- Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.
- Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

Problem 2: Show that the premises “Everyone in this discrete math class has taken a course in computer science” and “Melissa is a student in this discrete math class” lead to the conclusion “Melissa has taken a course in computer science.”

We can't always use formal proof techniques

Formal proofs are precise and "easy" for machines to construct…



… but are often tedious for humans to construct, interpret, or verify.

Result: Most mathematical proofs are actually constructed using informal proof techniques!

Another Example

- 1. It is not sunny this afternoon and it is colder than yesterday.
- 2. We will go swimming only if it is sunny.
- 3. If we do not go swimming then we will take a canoe trip.
- 4. If we take a canoe trip, then we will be home by sunset.

- Prove: We will be home by sunset.

What are the characteristics of an informal proof?

In an informal proof

- The statements making up the proof are typically not written in any formal language (e.g., propositional logic)
- Steps of the proof and derivations are often argued using English or mathematical formulas
- Multiple derivations may occur in a single step
- Axioms are often not all stated up front

As a result, it is sometimes easy to make mistakes writing informal proofs.

Final Thoughts

- Until today, we had look at representing different types of logical statements
- Rules of inference allow us to derive new results by reasoning about known truths
- Next lecture:
  - Proof techniques