Today's Topics

- Inclusion/exclusion principle
- The pigeonhole principle

Sometimes when counting a set, we count the same item more than once

For instance, if something can be done n_1 ways or n_2 ways, but some of the n_1 ways are the same as some of the n_2 ways.

In this case $n_1 + n_2$ is an overcount of the ways to complete the task!

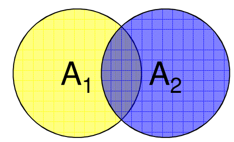
What we really want to do is count the $n_1 + n_2$ ways to complete the task and then subtract out the common solutions.

This is called the inclusion/exclusion principle.

We can formulate this concept using set theory

Suppose that a task T can be completed using a solution drawn from one of two classes: A_1 and A_2

As in the sum rule, we can define the solution set for the task T as $S = A_1 \cup A_2$



Then $|S| = |A_1 \cup A_2|$

$$= |A_1| + |A_2| - |A_1 \cap A_2|$$

Do you remember this from way back in the semester?

Counting Bit Strings

Example: How many bit strings of length 8 start with a 1 or end with 00?

Solution:

- $2^7 = 128$ 8-bit strings start with a 1
- $2^6 = 64$ 8-bit strings end with 00
- $2^5 = 32$ 8-bit strings start with a 1 and end with 00

So, we have $128 + 64 - 32 = 160$ ways to construct an 8-bit string that starts with a 1 or ends with 00.

Job Applications

Example: A company receives 350 applications. Suppose 220 of these people majored in CS, 147 majored in business, and 51 were double-majors. How many applicants majored in neither CS nor business?

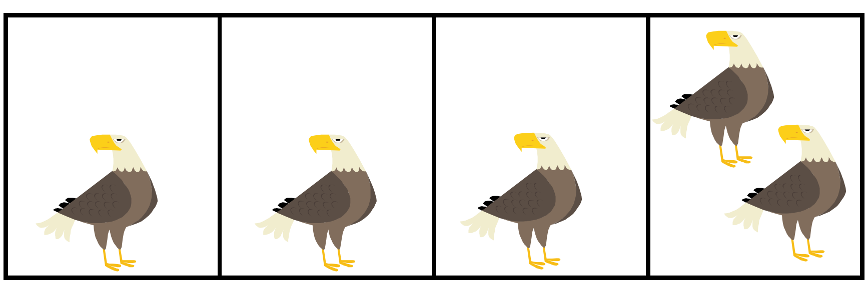
Solution:

- Let C be the set of CS majors, B be the set of business majors
- $|C \cup B| = |C| + |B| - |C \cap B|$
- $= 220 + 147 - 51$
- $= 316$

So of the 350 applications, 350 – 316 = 34 applications neither majored in CS nor business.

The pigeonhole principle is an incredibly simple concept that is extremely useful!

The pigeonhole principle: If k is a positive integer and k+1 objects are placed in k boxes, then at least one box contains at least two objects.



Example: k = 4

The pigeonhole principle is also easy to prove

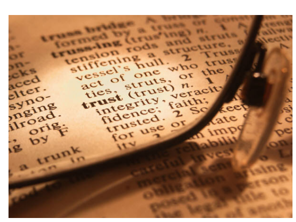
The pigeonhole principle: If k is a positive integer and k+1 objects are placed in k boxes, then at least one box contains at least two objects.

Proof: Assume that each of the k boxes contains at most 1 item. This means that there are at most k items, which is a contradiction of our assumption that we have k+1 items, so at least one box must contain more than one item. ❑

Examples

Example: Among any group of 367 people there are at least two with the same birthday, since there are only 366 possible birthdays.

Example: Among any 27 English words, at least two will start with the same letter.



Group Work!

Problem 1: How many bit strings of length 10 both begin and end with a 1?

Problem 2: How many bit strings of length 10 begin with three 0s or end with two 0s?

Problem 3: If a student can get either an A, B, C, D, or F on a test, how many students are needed to ensure that at least two get the same grade?

There is a more general form of the pigeonhole principle that is even more useful

The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ items.

Example

What is the minimum number of students needed such that at least six students receive the same grade, if possible grades are A, B, C, D, and F?

Solution:

- Need the smallest integer N such that $\lceil N/5 \rceil = 6$
- With 25 students, it would be possible (though unlikely) to have 6 students get each possible grade
- By adding a 26th student, we guarantee that at least 6 students get one possible grade
- So, the smallest such N is $5 \times 5 + 1 = 26$ ❑

From the casino…

How many cards must be drawn from a standard 52-card deck to guarantee that three cards of the same suit are drawn?

Solution:

- Let's make 4 piles: one for each suit
- We want to have $\lceil N/4 \rceil \geq 3$
- We can do this using $4 \times 2 + 1 = 9$ cards

Note: We don't need 9 cards to end up with three from the same suit---if we did, we could never get a flush in poker!

We can't always use the pigeonhole principle directly

How many card would we need to draw to ensure that we picked at least three hearts?

In the worst case, we would need to draw every club, spade, and diamond before getting three hearts…

So, to guarantee three hearts, we need to draw $3 \times 13 + 3 = 42$ cards!

Ma Bell…

What is the least number of area codes needed to guarantee that the 25 million phones in some state can be assigned distinct 10-digit phone numbers of the form NXX-NXX-XXXX?

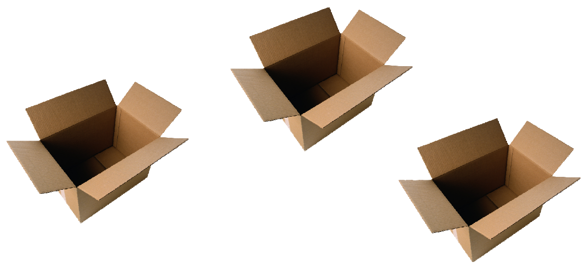
Solution:

- The product rule tells us that there are 8 million phone numbers of the form NXX-XXXX
- Think of phones as objects and phone numbers as boxes
- By the generalized pigeonhole principle, we know that some "box" contains at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil = 4$ "objects"
- This means that we need 4 area codes to ensure that each phone gets a unique 10-digit number ❑

This has been easy so far, right?

Unfortunately, life isn't always easy!

Sometimes, we need to be clever when we are defining our "boxes" or assigning objects to them



For example…

Sports…

During a month with 30 days, a baseball team plays at least one game per day, but no more than 45 games total. Show that there must be some period of consecutive days in which exactly 14 games are played.

Solution:

- Let a_j be the number of games played on or before the j^{th} day of the month. Note that the sequence $\{a_j\}$ is strictly increasing.
- Note also that $\{a_j + 14\}$ is also an increasing sequence
- Now, consider $a_1, a_2, …, a_{30}, a_1 + 14, a_2 + 14, …, a_{30} + 14$
- There are 60 terms in this sequence, all $\leq (45 + 14) = 59$
- By the pigeonhole principle, at least two terms are equal
- Note: Each a_j for $j = 1, 2, …, 30$ is distinct, as is each $a_j + 14$
- This means there exists some a_i that is equal to some $a_j + 14$, so 14 games were played from day $j + 1$ to day i ❑

Group Work!

Problem 1: What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

Problem 2: A drawer contains a dozen brown socks and a dozen black socks, all unmatched. How many socks must be drawn to find a matching pair? How many socks must be drawn to find a pair of black socks?

Final Thoughts

- The inclusion/exclusion principle is useful when we need to avoid overcounting
- The pigeonhole principle and its generalized form are useful for solving many types of counting problems