Today's Topics

- Introduction to combinatorics
- Product rule
- Sum rule

What *is* combinatorics?

Combinatorics is the study of arrangements of discrete objects.

Just think of this as a fancy word for "counting"

Many applications throughout computer science:

- Algorithm complexity analysis
- Resource allocation & scheduling
- Security analysis
- …

Today, we will learn the basics of counting. More advanced topics will be covered in later lectures.

A motivating example…

To access most computer systems, you need to login with a user name and a password.

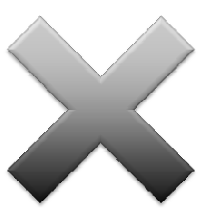
Suppose that for a certain system

- Passwords must contain either 6, 7, or 8 characters
- Each character must be an uppercase letter or a digit
- Every password must contain at least one digit

How many valid passwords are there?

Solving these types of problems requires that we learn how to count complex objects

Fortunately, we can solve many types of combinatorial problems using two simple rules:



The product rule

The sum rule

Product rule applies when a counting problem can be broken into multiple tasks

The Product Rule: Suppose a procedure can be broken into a sequence $t_1, t_2, \ldots, t_k$ of tasks. Further, let there be $n_1, n_2, \ldots, n_k$ ways to complete each task. Then there are $n_1 \times n_2 \times \ldots \times n_k$ ways to complete the procedure.

To apply the product rule, do the following:

1. Identify each task $t_1, \ldots, t_k$
2. For each task t_i, determine the n_i, the number of possible ways to complete t_i
3. Compute $n_1 \times n_2 \times \ldots \times n_k$

Let's look at a few examples…

An easy example: Assigning offices

Example: It is Alice's and Bob's first day of work at Acme, Inc. If there are 10 unused offices at Acme, how many ways can Alice and Bob be assigned an office?

Step 1: Determine tasks

1. Give Alice an office
2. Give Bob an office

Step 2: Count possible completions

1. Can give any one of 10 offices to Alice
2. Can give any one of the remaining 9 offices to Bob

Step 3: Compute the product

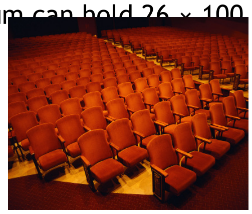
- Alice and Bob can be assigned offices in $10 \times 9 = 90$ ways!

Auditorium Seating

Example: The chairs in an auditorium are to be labeled using an upper case letter and a positive number not exceeding 100 (i.e., B23). What is the maximum number of seats that can be placed in the auditorium?

Solution:

- Task 1: Count the letters that can be used
- Task 2: Count the numbers (100)
- So, the auditorium can hold 26 × 100 = 2600 chairs.



Counting Bit Strings

Example: How many bit strings of length 5 are there?

Solution:

- Task 1: Choose first bit
- Task 2: Choose second bit (2)
- Task 3: Choose third bit (2)
- Task 4: Choose fourth bit (2)
- Task 5: Choose fifth bit (2)

- So, there are $2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$ bit strings of length 5

License Plates

Example: Suppose that in some state, license plates consist of three letters followed by three decimal digits. How many valid license plates are there?

$$\underbrace{\underline{A}\ \underline{B}\ \underline{C}}_{\text{26 choices for each}}\ \underbrace{\underline{1}\ \underline{2}\ \underline{3}}_{\text{10 choices for each}}$$

Solution: There are $26^3 \times 10^3 = 17{,}576{,}000$ possible valid license plates

Group Work!

Problem 1: How many different three-letter initials can people have?

Problem 2: There are 18 mathematics major and 325 CS majors. How many ways are there to pick two people so that one is a math major and the other is a CS major.

The sum rule applies when a single task can be completed using several different approaches

The Sum Rule: Suppose that a single task can be completed in either one of n_1 ways, one of n_2 ways, …, or one of n_k ways. Then the task can be completed in $n_1 + n_2 + \ldots + n_k$ different ways.

Note: We can break the set of all possible solutions to the problem into disjoint subsets. E.g., if we have k "classes" of solutions, then $S = S_1 \cup S_2 \cup \ldots \cup S_k$

- $|S| = |S_1 \cup S_2 \cup \ldots \cup S_k|$
- $= |S_1| + |S_2| + \ldots + |S_k|$ Since $S_1, \ldots, S_K$ are disjoint
- $= n_1 + n_2 + \ldots + n_k$

University Committees

Example: Suppose that either a CS professor or a CS graduate student can be nominated to serve on a particular university committee. If there are 21 CS professors and 101 CS graduate students, how many ways can this seat on the committee be chosen?

Solution:

- Let
  - P be the set of professors
  - G be the set of graduate students
  - S be the solution set, with $S = P \cup G$
- Then there are $|S| = |P \cup G| = |P| + |G| = 21 + 101 = 122$ ways to fill the empty seat on the committee.

Travel Choices

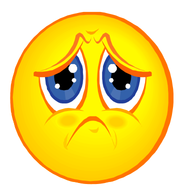
Example: Jane wants to travel from Pittsburgh to New York City. If she flies, she can leave at any one of 12 departure times. If she takes the bus, she can leave at any one of 6 departure times. If she takes the train, she can leave at any one of 4 departure times. How many different departure times can Jane choose from?

Solution:

- $S = F \cup B \cup T$, so
- $|S| = |F \cup B \cup T|$
- $= |F| + |B| + |T|$
- $= 12 + 6 + 4$
- $= 22$ departure times

The product and sum rules are kind of boring…

Most interesting counting problems cannot be solved using the product rule or the sum rule alone…



… but many interesting problems can be solved by combining these two approaches!

Let's revisit our password example…

Passwords revisited…

To access most computer systems, you need to login with a user name and a password.

Use the product rule to count passwords of each possible length!

Choices: Sum rule!

Suppose that for a certain system
- Passwords must contain either 6, 7, or 8 characters
- Each character must be an uppercase letter or a digit
- Every password must contain at least one digit

How many valid passwords are there?

First, we'll apply the sum rule

Let:
- P_6 = Set of passwords of length 6
- P_7 = Set of passwords of length 7
- P_8 = Set of passwords of length 8
- $S = P_6 \cup P_7 \cup P_8$

Note: $|S| = |P_6| + |P_7| + |P_8|$

Since each element of P_6, P_7, and P_8 is made up of independent choices of letters and numbers, we can apply the product rule to determine $|P_6|$, $|P_7|$, and $|P_8|$

Recall: a password must contain at least one number!

Observation: To figure out the number of 6-character passwords containing at least one number, it is easier for us to count all 6-character passwords and then subtract away those passwords not containing a number.

Note: there are

- $(26 + 10)^6 = 36^6$ 6-character passwords
- 26^6 6-character passwords not containing a digit

So, $|P_6| = 36^6 - 26^6 = 1{,}867{,}866{,}560$

Wrapping it all up…

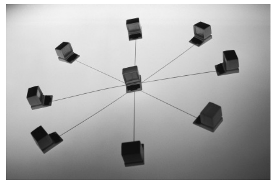
We can compute

- $|P_6| = 36^6 - 26^6 = 1{,}867{,}866{,}560$
- $|P_7| = 36^7 - 26^7 = 70{,}332{,}353{,}920$
- $|P_8| = 36^8 - 26^8 = 208{,}827{,}064{,}576$

By leveraging our earlier observation that $|S| = |P_6| + |P_7| + |P_8|$, we can conclude that there are 2,612,282,842,880 valid passwords for our target system.

IP Addresses

An IP address is a 32-bit string that is used to identify a computer that is connected to the Internet.



There are three categories of IP addresses that can be assigned to computers:

1. Class A addresses consist of the prefix "0" followed by a 7-bit network ID and a 24-bit host ID
2. Class B addresses consist of the prefix "10" followed by a 14-bit network ID and a 16-bit host ID
3. Class C addresses consist of the prefix "110" followed by a 21-bit network ID and an 8-bit host ID

So how many valid IP addresses are there?

Note: IP addresses are subject to restrictions:

- 1111111 cannot be used as the network ID of a Class A IP
- Host IDs consisting of only 1s or only 0s cannot be used

To count IP addresses, we will use the sum rule and the product rule. So $S = S_A \cup S_B \cup S_C$, so $|S| = |S_A| + |S_B| + |S_C|$

Compute S_A:

- $2^7 - 1$ network IDs since 1111111 can't be used
- $2^{24} - 2$ host IDs for each network ID
- Total of 2,130,706,178 Class A IP addresses

So how many valid IP addresses are there? (cont.)

Compute S_B:

- 2^{14} network IDs
- 2^{16} – 2 host IDs for each network ID
- Total of 1,073,709,056 Class B IP addresses

Compute S_C:

- 2^{21} network IDs
- 2^{8} - 2 host IDs for each network ID
- Total of 532,676,608 Class C IP addresses

Since $|S| = |S_A| + |S_B| + |S_C|$, there are 3,737,091,842 IP addresses that can be assigned to computers connected to the Internet!

Group Work!

Problem 1: A committee is formed by choosing one representative from each of the 50 US states. This representative is either the governor of that state, or one of the two senators from that state. How many possible ways are there to form this committee?

Problem 2: How many license plates can be made using either two letters followed by four digits or two digits followed by four letters?

Final Thoughts

- Combinatorics is just a fancy word for counting!
- There are many any uses of combinatorics throughout computer science
- We can solve a variety of interesting problems using simple rules like the product rule and the sum rule