CS441 – Discrete Structures for Computer Science
Instructor: Dr.Litman

Problem from Section 3.5

10. We must find, by inspection with mental arithmetic, the greatest common divisors of the numbers from 1 to 11 with 12, and list those whose gcd is 1. These are 1, 5, 7, and 11. There are so few since 12 had many factors—in particular, both 2 and 3.

12. Since these numbers are small, the easiest approach is to find the prime factorization of each number and look for any common prime factors.

a) Since $21 = 3 \cdot 7$, $34 = 2 \cdot 17$, and $55 = 5 \cdot 11$, these are pairwise relatively prime.

b) Since $85 = 5 \cdot 17$, these are not pairwise relatively prime.

c) Since $25 = 5^2$, 41 is prime, $49 = 7^2$, and $64 = 2^6$, these are pairwise relatively prime.

d) Since 17, 19, and 23 are prime and $18 = 2 \cdot 3^2$, these are pairwise relatively prime.

20. We form the greatest common divisors by finding the minimum exponent for each prime factor.

a) $2^2 \cdot 3^3 \cdot 5^2$ b) $2 \cdot 3 \cdot 11$ c) 17 d) 1 e) 5 f) $2 \cdot 3 \cdot 5 \cdot 7$

22. We form the least common multiples by finding the maximum exponent for each prime factor.

a) $2^5 \cdot 3^3 \cdot 5^5$ b) $2^{11} \cdot 3^9 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17^{14}$ c) 17^{17} d) $2^2 \cdot 5^3 \cdot 7 \cdot 13$

e) undefined (0 is not a positive integer) f) $2 \cdot 3 \cdot 5 \cdot 7$

Problem from Section 4.1

4. a) Plugging in $n = 1$ we have that $P(1)$ is the statement $1^3 = [1 \cdot (1+1)/2]^2$.

b) Both sides of $P(1)$ shown in part (a) equal 1.

c) The inductive hypothesis is the statement that

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 .$$

d) For the inductive step, we want to show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can prove

$$[1^3 + 2^3 + \cdots + k^3] + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 .$$

e) Replacing the quantity in brackets on the left-hand side of part (d) by what it equals by virtue of the inductive hypothesis, we have

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + k + 1\right) = (k+1)^2\left(\frac{k^2+4k+4}{4}\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2 ,$$

as desired.

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

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Problem from Section 4.1

18. a) Plugging in $n = 2$, we see that $P(2)$ is the statement $2! < 2^2$.
b) Since $2! = 2$, this is the true statement $2 < 4$.
c) The inductive hypothesis is the statement that $k! < k^k$.
d) For the inductive step, we want to show for each $k \geq 2$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can prove that $(k+1)! < (k+1)^{k+1}$.
e) $(k+1)! = (k+1)k! < (k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}$
f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n greater than 1.

32. The statement is true for the base case, $n = 0$, since $3 \mid 0$. Suppose that $3 \mid (k^3 + 2k)$. We must show that $3 \mid ((k+1)^3 + 2(k+1))$. If we expand the expression in question, we obtain $k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)$. By the inductive hypothesis, 3 divides $k^3 + 2k$, and certainly 3 divides $3(k^2 + k + 1)$, so 3 divides their sum, and we are done.