Uncertainty

Chapter 13

Outline

- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule

Uncertainty

Let action A_t = leave for airport $_t$ minutes before flight
Will A_t get me there on time?

Problems:

1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports)
3. uncertainty in action outcomes (flat tire, etc.)
4. immense complexity of modeling and predicting traffic

Hence a purely logical approach either

1. risks falsehood: "A_{25} will get me there on time", or
2. leads to conclusions that are too weak for decision making:

"A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

(A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport …)

But…

- A decision must be made!
- No intelligent system can afford to consider all eventualities, wait until all the data is in and complete, or try all possibilities to see what happens

Quick Overview of Reasoning Systems

- Logic:True or false, nothing in between. No uncertainty
- Non-monotonic logic:True or false, but new information can change it.
- Probability:Degree of belief, but in the end it’s either true or false
- Fuzzy:Degree of belief, allows overlapping of true and false states

Examples

- Logic: All birds fly
- Non-monotonic
  - Tweety flies, since he’s a bird and no evidence he doesn’t fly

Probability

Probabilistic assertions summarize effects of
- laziness: failure to enumerate exceptions, qualifications, etc.
- ignorance: lack of relevant facts, initial conditions, etc.

Subjective probability:
- Probabilities relate propositions to agent's own state of knowledge

  e.g., $P(A_{25} \mid \text{no reported accidents}) = 0.06$

These are not assertions about the world

Probabilities of propositions change with new evidence:

e.g., $P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$

Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time $\mid \ldots) = 0.04$

$P(A_{90}$ gets me there on time $\mid \ldots) = 0.70$

$P(A_{120}$ gets me there on time $\mid \ldots) = 0.95$

$P(A_{1440}$ gets me there on time $\mid \ldots) = 0.9999$

- Which action to choose?

  Depends on my preferences for missing flight vs. time spent waiting, etc.
  - Utility theory is used to represent and infer preferences
  - Decision theory = probability theory + utility theory

Syntax

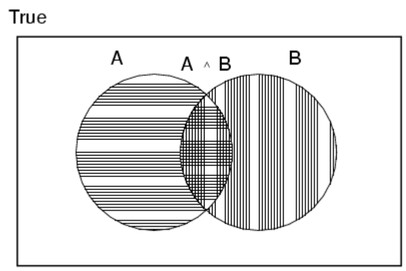
- Basic element: random variable
- Similar to propositional logic: possible worlds defined by assignment of values to random variables.
- Boolean random variables
  e.g., *Cavity* (do I have a cavity?)
- Discrete random variables
  e.g., *Weather* is one of <*sunny,rainy,cloudy,snow*>
- Domain values must be exhaustive and mutually exclusive
- Elementary proposition constructed by assignment of a value to a random variable: e.g., *Weather = sunny*, *Cavity = false* (abbreviated as $\neg$*cavity*)
- Complex propositions formed from elementary propositions and standard logical connectives e.g., *Weather = sunny* $\vee$ *Cavity = false*

Syntax

- Atomic event: A complete specification of the state of the world about which the agent is uncertain
  E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:
  Cavity = false $\wedge$ *Toothache = false*
  Cavity = false $\wedge$ *Toothache = true*
  Cavity = true $\wedge$ *Toothache = false*
  Cavity = true $\wedge$ *Toothache = true*
- Atomic events are mutually exclusive and exhaustive

Axioms of probability

- For any propositions *A, B*
  - $0 \leq P(A) \leq 1$
  - $P(true) = 1$ and $P(false) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$



Prior probability

- Prior or unconditional probabilities of propositions
  e.g., P(*Cavity* = true) = 0.1 and P(*Weather* = sunny) = 0.72 correspond to belief prior to arrival of any (new) evidence
- Probability distribution gives values for all possible assignments:
  P(*Weather*) = <0.72,0.1,0.08,0.1> (normalized, i.e., sums to 1)
- Joint probability distribution for a set of random variables gives the probability of every atomic event on those random variables
  P(*Weather,Cavity*) = a 4 × 2 matrix of values:

Weather =	sunny	rainy	cloudy	snow
Cavity = true	0.144	0.02	0.016	0.02
Cavity = false	0.576	0.08	0.064	0.08

- Every question about a domain can be answered by the joint distribution

How could we estimate the full joint distribution?

Parameter estimates are provided by expert knowledge, statistics on data samples, or a combination of both.

Suppose you have 20 variables.

Expert knowledge:

P(X1=0,X2=0,…,X13=1,…,X20=0) vs.

P(X1=0,X2=0,…,X13=0,…,X20=0) ?

Data Samples: practically speaking, we don't typically have enough data

Conditional probability

- Conditional or posterior probabilities
  e.g., P(*cavity* | *toothache*) = 0.8
  i.e., given that *toothache* is all I know
- (Notation for conditional distributions:
  P(*Cavity* | *Toothache*) = 2-element vector of 2-element vectors)
- If we know more, e.g., *cavity* is also given, then we have
  P(*cavity* | *toothache,cavity*) = 1
- New evidence may be irrelevant, allowing simplification, e.g.,
  P(*cavity* | *toothache, sunny*) = P(*cavity* | *toothache*) = 0.8
- This kind of inference, sanctioned by domain knowledge, is crucial

More on Conditional Probabilities

- P (CarWontStart | NoGas)
  - This predicts a symptom based on an underlying cause
  - These can be generated empirically (Drain N gastanks, see how many cars start) or using expert knowledge
- P (NoGas | CarWontStart)
  - Diagnosis. We have a symptom and want to predict the cause. This is what the system wants to determine

Conditional probability

- Definition of conditional probability:
  $P(a \mid b) = P(a \wedge b) / P(b)$ if $P(b) > 0$
- Product rule gives an alternative formulation:
  $P(a \wedge b) = P(a \mid b) P(b) = P(b \mid a) P(a)$
- A general version holds for whole distributions, e.g.,
  P(*Weather,Cavity*) = **P**(*Weather* | *Cavity*) **P**(*Cavity*)
- (View as a set of 4 × 2 equations, not matrix mult.)
- Chain rule is derived by successive application of product rule:
  $\mathbf{P}(X_1, \ldots, X_n) = \mathbf{P}(X_1,\ldots,X_{n-1})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1})$
  $= \mathbf{P}(X_1,\ldots,X_{n-2})\, \mathbf{P}(X_{n-1} \mid X_1,\ldots,X_{n-2})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1})$
  $= \ldots$
  $= \pi_{i=1}$^n $\mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})$

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$

Inference by enumeration

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	toothache		*¬ toothache*	
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- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2

Inference by enumeration

- Start with the joint probability distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- Can also compute conditional probabilities:

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016+0.064}{0.108 + 0.012 + 0.016 + 0.064}$$

$$= 0.4$$

Normalization

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- Denominator can be viewed as a normalization constant α

P(*Cavity* | *toothache*) = α, **P**(*Cavity,toothache*)

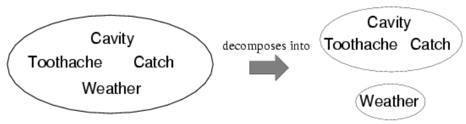
= α, [**P**(*Cavity,toothache,catch*) + **P**(*Cavity,toothache,*¬ *catch*)]

= α, [<0.108,0.016> + <0.012,0.064>]

= α, <0.12,0.08> = <0.6,0.4>

Independence

- *A* and *B* are independent iff
  P(*A*|*B*) = **P**(*A*) or **P**(*B*|*A*) = **P**(*B*) or **P**(A, B) = **P**(*A*) **P**(*B*)



P(*Toothache, Catch, Cavity, Weather*)
= **P**(*Toothache, Catch, Cavity*) **P**(*Weather*)

- 32 entries reduced to 12; for *n* independent biased coins, $O(2^n) \to O(n)$
- Absolute independence powerful but rare
- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  (1) **P**(*catch* | *toothache, cavity*) = **P**(*catch* | *cavity*)
- The same independence holds if I haven't got a cavity:
  (2) **P**(*catch* | *toothache,¬cavity*) = **P**(*catch* | *¬cavity*)
- *Catch* is conditionally independent of *Toothache* given *Cavity*:
  P(*Catch* | *Toothache,Cavity*) = **P**(*Catch* | *Cavity*)
- Equivalent statements:
  P(*Toothache* | *Catch, Cavity*) = **P**(*Toothache* | *Cavity*)
  P(*Toothache, Catch* | *Cavity*) = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*)

Conditional independence contd.

- Write out full joint distribution using chain rule:
  P(*Toothache, Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch* | *Cavity*) **P**(*Cavity*)
  = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*) **P**(Cavity)

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*.
- Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

- Product rule $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$
  $\Rightarrow$ Bayes' rule: $P(a \mid b) = P(b \mid a)\, P(a) / P(b)$
- or in distribution form
  $\mathbf{P}(Y|X) = \mathbf{P}(X|Y)\, \mathbf{P}(Y) / \mathbf{P}(X) = \alpha \mathbf{P}(X|Y)\, \mathbf{P}(Y)$
- Useful for assessing diagnostic probability from causal probability:
  - P(Cause|Effect) = P(Effect|Cause) P(Cause) / P(Effect)
  - E.g., let *M* be meningitis, *S* be stiff neck:
    $P(m|s) = P(s|m)\, P(m) / P(s) = 0.8 \times 0.0001 / 0.1 = 0.0008$
  - Note: posterior probability of meningitis still very small!

Bayes' Rule and conditional independence

P(*Cavity* | *toothache* ∧ *catch*)

= α**P**(*toothache* ∧ *catch* | *Cavity*) **P**(*Cavity*)

= α**P**(*toothache* | *Cavity*) **P**(*catch* | *Cavity*) **P**(*Cavity*)

- This is an example of a naïve Bayes model:

  $\mathbf{P}(\text{Cause},\text{Effect}_1, \ldots ,\text{Effect}_n) = \mathbf{P}(\text{Cause})\ \pi_i \mathbf{P}(\text{Effect}_i|\text{Cause})$

- Total number of parameters is linear in *n*
- All features/symptoms/effects conditionally independent of each other given the class/diagnosis/cause

Summary

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- Independence and conditional independence provide the tools