Propositional Logic

Chapter 7

Outline

- Review
  - Knowledge-based agents
  - Logic in general
  - Propositional logic in particular – syntax and semantics
- Wumpus world
- Inference rules and theorem proving
  - Resolution
  - forward chaining
  - backward chaining

Logic in general

- Logics are formal languages for representing information such that conclusions can be drawn
- Syntax defines the sentences in the language
- Semantics define the "meaning" of sentences;
  – i.e., define truth of a sentence in a world

- E.g., the language of arithmetic
  – $x+2 \geq y$ is a sentence; $x2+y > \{\}$ is not a sentence
  – $x+2 \geq y$ is true iff the number $x+2$ is no less than the number y
  – $x+2 \geq y$ is true in a world where $x = 7$, $y = 1$
  – $x+2 \geq y$ is false in a world where $x = 0$, $y = 6$

Entailment

- Entailment means that one thing follows from another:

$$KB \models \alpha$$

- Knowledge base *KB* entails sentence α if and only if α is true in all worlds where *KB* is true

  – E.g., the KB containing "the Steelers won" and "the Bengals won" entails "Either the Steelers won or the Bengals won"

  – E.g., $x+y = 4$ entails $4 = x+y$

  – Entailment is a relationship between sentences (i.e., syntax) that is based on semantics

A	B	C	A ∧ B	A ∧ C	B ∧ C
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	T	T	F	F	T
T	F	F	F	F	F
T	F	T	F	T	F
T	T	F	T	F	F
T	T	T	T	T	T

A^C, C does not entail B∧C

A,B, Entails A∧B

Inference

- $KB \vdash_i \alpha$ = sentence α can be derived from *KB* by procedure *i*
- Soundness: *i* is sound if whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$
- Completeness: *i* is complete if whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$
- Preview: we will define a logic (first-order logic) which is expressive enough to say almost anything of interest, and for which there exists a sound and complete inference procedure.
- That is, the procedure will answer any question whose answer follows from what is known by the *KB*.

Propositional logic: Syntax

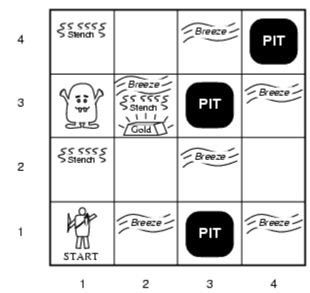
- Propositional logic is the simplest logic – illustrates basic ideas
- The proposition symbols P_1, P_2 etc are sentences
  - If S is a sentence, $\neg S$ is a sentence (negation)
  - If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)
  - If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)
  - If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)
  - If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Propositional Logic: Semantics (truth tables for connectives)

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

Wumpus World PEAS description

- Performance measure
  - gold +1000, death -1000
  - -1 per step, -10 for using the arrow
- Environment
  - Squares adjacent to wumpus are smelly
  - Squares adjacent to pit are breezy
  - Glitter iff gold is in the same square
  - Shooting kills wumpus if you are facing it
  - Shooting uses up the only arrow
  - Grabbing picks up gold if in same square
  - Releasing drops the gold in same square



- Sensors: Stench, Breeze, Glitter, Bump, Scream
- Actuators: Left turn, Right turn, Forward, Grab, Release, Shoot

Wumpus world characterization

- Fully Observable
- Deterministic
- Episodic
- Static
- Discrete
- Single-agent?

Wumpus world characterization

- Fully Observable No – only local perception
- Deterministic Yes – outcomes exactly specified
- Episodic No – sequential at the level of actions
- Static Yes – Wumpus and Pits do not move
- Discrete Yes
- Single-agent? Yes – Wumpus is essentially a natural feature

Wumpus World continued

- Main difficulty: Agent doesn't know the configuration
- Reason about configuration
- Knowledge evolves as new percepts arrive and actions are taken.

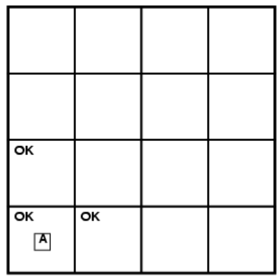
Wumpus Example

		stench	[Wumpus]	stench
		Glitter [gold]	stench, breeze	
0	[start]	breeze	[Pit]	breeze
	0			

Examples of reasoning

- If the player is in square (1, 0) and the percept is breeze, then there must be a pit in (0,0) or a pit in (2,0) or a pit in (1,1).
- If the player is now in (0,0) [and still alive], there is not a pit in (0,0).
- If there is no breeze percept in (0,0), there is no pit in (0,1)
- If there is also no breeze in (0,1) then there is no pit in (1,1).
- Therefore, there must be a pit in (2,0)

Exploring a wumpus world



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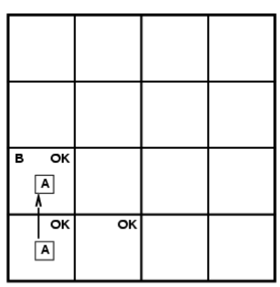

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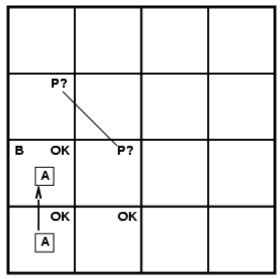

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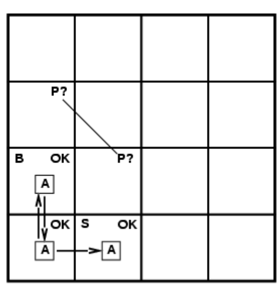

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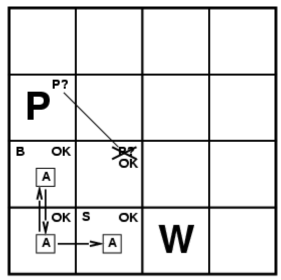

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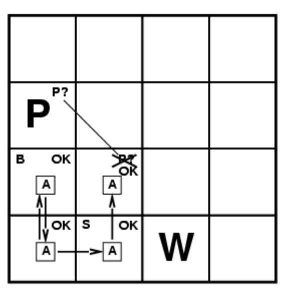

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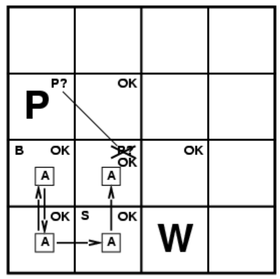

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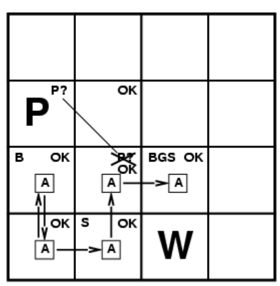
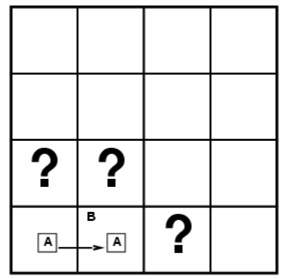


Entailment in the wumpus world

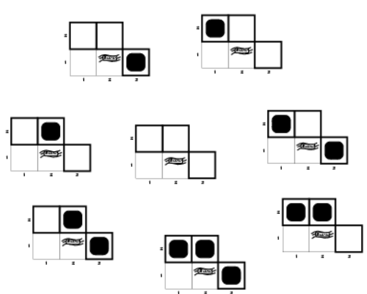
Situation after detecting nothing in [1,1], moving right, breeze in [2,1]

Consider possible models for *KB* assuming only pits

3 Boolean choices ⇒ 8 possible models



Wumpus models



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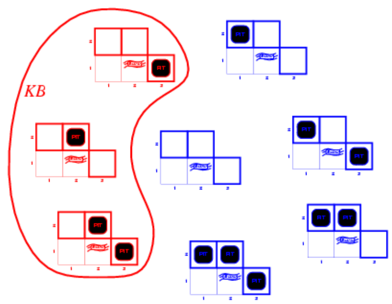
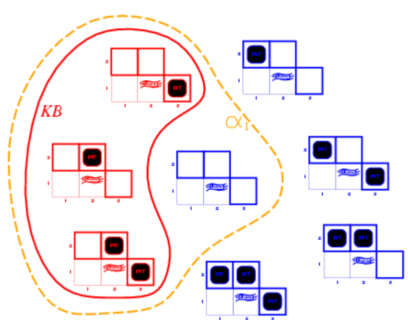


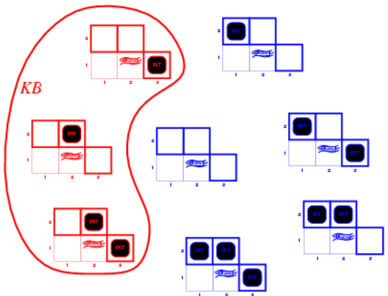
- *KB* = wumpus-world rules + observations

Wumpus models



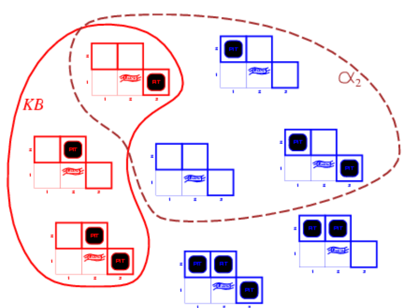
- *KB* = wumpus-world rules + observations
- α_1 = "[1,2] is safe", *KB* $\models \alpha_1$, proved by model checking

Wumpus models



- *KB* = wumpus-world rules + observations

Wumpus models



- *KB* = wumpus-world rules + observations
- α_2 = "[2,2] is safe", *KB* $\not\models \alpha_2$

Logical Representation of Wumpus

Is there a pit in [i, j]?
Is there a breeze in [i, j]?

Pits cause breezes in adjacent squares.

Some Wumpus world sentences

Let $P_{i,j}$ be true if there is a pit in [i, j].
Let $B_{i,j}$ be true if there is a breeze in [i, j].

$\neg P_{1,1}$
$\neg B_{1,1}$
$B_{2,1}$
...

- "Pits cause breezes in adjacent squares"

$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$
$B_{2,1} \Leftrightarrow (P_{1,1} \lor P_{2,2} \lor P_{3,1})$
…

Inference-based agents in the wumpus world

A wumpus-world agent using propositional logic:

$\neg P_{1,1}$
$\neg W_{1,1}$
$B_{x,y} \Leftrightarrow (P_{x,y+1} \lor P_{x,y-1} \lor P_{x+1,y} \lor P_{x-1,y})$
$S_{x,y} \Leftrightarrow (W_{x,y+1} \lor W_{x,y-1} \lor W_{x+1,y} \lor W_{x-1,y})$
$W_{1,1} \lor W_{1,2} \lor \ldots \lor W_{4,4}$
$\neg W_{1,1} \lor \neg W_{1,2}$
$\neg W_{1,1} \lor \neg W_{1,3}$
…

$\Rightarrow$ 64 distinct proposition symbols, 155 sentences

Truth tables for inference

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	KB	α_1
false	*false*	*false*	*false*	*false*	*false*	*false*	*false*	*true*
false	*false*	*false*	*false*	*false*	*false*	*true*	*false*	*true*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
false	*true*	*false*	*false*	*false*	*false*	*false*	*false*	*true*
false	*true*	*false*	*false*	*false*	*false*	*true*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*false*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*true*	<u>*true*</u>	<u>*true*</u>
false	*true*	*false*	*false*	*true*	*false*	*false*	*false*	*true*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
true	*true*	*true*	*true*	*true*	*true*	*true*	*false*	*false*

Inference by enumeration

- Depth-first enumeration of all models is sound and complete
- For *n* symbols, time complexity is $O(2^n)$, space complexity is $O(n)$

Logical equivalence

- Two sentences are logically equivalent iff true in same models: $\alpha \equiv \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) \quad \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) \quad \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) \quad \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) \quad \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) \quad \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) \quad \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \quad \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

Example Proof by Deduction

- Knowledge

  S1: $B_{22} \Leftrightarrow (P_{21} \vee P_{23} \vee P_{12} \vee P_{32})$ *rule*

  S2: $\neg B_{22}$ *observation*

- Inferences

  S3: $(B_{22} \Rightarrow (P_{21} \vee P_{23} \vee P_{12} \vee P_{32})) \wedge$
  $((P_{21} \vee P_{23} \vee P_{12} \vee P_{32}) \Rightarrow B_{22})$ *S1,bi elim*

  S4:

  S5:

  S6:

  S7:

Example Proof by Deduction

- Knowledge

  S1: $B_{22} \Leftrightarrow (P_{21} \vee P_{23} \vee P_{12} \vee P_{32})$ *rule*

  S2: $\neg B_{22}$ *observation*

- Inferences

  S3: $(B_{22} \Rightarrow (P_{21} \vee P_{23} \vee P_{12} \vee P_{32})) \wedge$
  $((P_{21} \vee P_{23} \vee P_{12} \vee P_{32}) \Rightarrow B_{22})$ *S1,bi elim*

  S4: $((P_{21} \vee P_{23} \vee P_{12} \vee P_{32}) \Rightarrow B_{22})$ *S3, and elim*

  S5: $(\neg B_{22} \Rightarrow \neg(P_{21} \vee P_{23} \vee P_{12} \vee P_{32}))$ *contrapos*

  S6: $\neg(P_{21} \vee P_{23} \vee P_{12} \vee P_{32})$ *S2,S5, MP*

  S7: $\neg P_{21} \wedge \neg P_{23} \wedge \neg P_{12} \wedge \neg P_{32}$ *S6, DeMorg*

Proof methods

- Proof methods divide into (roughly) two kinds:
  - Application of inference rules
    - Legitimate (sound) generation of new sentences from old
    - Proof = a sequence of inference rule applications
      Can use inference rules as operators in a standard search
    - Typically require transformation of sentences into a normal form
  - Model checking
    - truth table enumeration (always exponential in n)
    - improved backtracking, e.g., Davis--Putnam-Logemann-Loveland (DPLL)
    - heuristic search in model space (sound but incomplete)
      e.g., min-conflicts-like hill-climbing algorithms

Resolution

Conjunctive Normal Form (CNF)

conjunction of disjunctions of literals

clauses

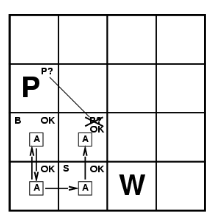
E.g., $(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$

- Resolution inference rule (for CNF):

$$\frac{\ell_i \lor \ldots \lor \ell_k, \qquad m_1 \lor \ldots \lor m_n}{\ell_i \lor \ldots \lor \ell_{i-1} \lor \ell_{i+1} \lor \ldots \lor \ell_k \lor m_1 \lor \ldots \lor m_{j-1} \lor m_{j+1} \lor \ldots \lor m_n}$$

where ℓ_i and m_j are complementary literals.

E.g., $\dfrac{P_{1,3} \lor P_{2,2}, \qquad \neg P_{2,2}}{P_{1,3}}$



- Resolution is sound and complete for propositional logic

Resolution in Wumpus World

- There is a pit at 2,1 or 2,3 or 1,2 or 3,2
  - $P_{21} \vee P_{23} \vee P_{12} \vee P_{32}$
- There is no pit at 2,1
  - $\neg P_{21}$
- Therefore (by resolution) the pit must be at 2,3 or 1,2 or 3,2
  - $P_{23} \vee P_{12} \vee P_{32}$

Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.

3. Move $\neg$ inwards using de Morgan's rules and double-negation:

4. Apply distributivity law ($\wedge$ over $\vee$) and flatten:

Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
   $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.
   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

3. Move $\neg$ inwards using de Morgan's rules and double-negation:
   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$

4. Apply distributivity law ($\wedge$ over $\vee$) and flatten:
   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

$B_{22} \Leftrightarrow (P_{21} \vee P_{23} \vee P_{12} \vee P_{32})$

1. Eliminate $\Leftrightarrow$, replacing with two implications
   $(B_{22} \Rightarrow (P_{21} \vee P_{23} \vee P_{12} \vee P_{32})) \wedge ((P_{21} \vee P_{23} \vee P_{12} \vee P_{32}) \Rightarrow B_{22})$
2. Replace implication $(A \Rightarrow B)$ by $\neg A \vee B$
   $(\neg B_{22} \vee (P_{21} \vee P_{23} \vee P_{12} \vee P_{32})) \wedge (\neg(P_{21} \vee P_{23} \vee P_{12} \vee P_{32}) \vee B_{22})$
3. Move $\neg$ "inwards" (unnecessary parens removed)
   $(\neg B_{22} \vee P_{21} \vee P_{23} \vee P_{12} \vee P_{32}) \wedge ((\neg P_{21} \wedge \neg P_{23} \wedge \neg P_{12} \wedge \neg P_{32}) \vee B_{22})$
4. Distributive Law
   $(\neg B_{22} \vee P_{21} \vee P_{23} \vee P_{12} \vee P_{32}) \wedge (\neg P_{21} \vee B_{22}) \wedge (\neg P_{23} \vee B_{22}) \wedge (\neg P_{12} \vee B_{22}) \wedge (\neg P_{32} \vee B_{22})$

Last Step

- Sentences are now in CNF:
- (P1 v P2 v ~P3) ^ P4 ^ ~P5 ^ (P2 v P3)
- Create a separate clause corresponding to each conjunct
  - P1 v P2 v ~P3
  - P4
  - ~P5
  - P2 v P3

Simple Resolution Example

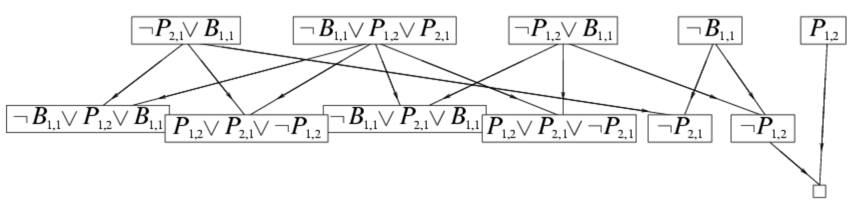
- When the agent is in 1,1, there is no breeze, so there can be no pits in neighboring squares
- Percept: ~B11
- Prove: ~P12.

Resolution example

- *KB* = $(B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$
- $\alpha = \neg P_{1,2}$

Resolution example

- *KB* = $(B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$
- $\alpha = \neg P_{1,2}$



Forward and backward chaining

- Horn Form (restricted)

  KB = conjunction of Horn clauses

  - Horn clause =
    - proposition symbol; or
    - (conjunction of symbols) ⇒ symbol
  - E.g., $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$
- Modus Ponens (for Horn Form): complete for Horn KBs

$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \ldots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

- Can be used with forward chaining or backward chaining.
- These algorithms are very natural and run in linear time

Forward chaining

- Idea: fire any rule whose premises are satisfied in the *KB*,
  - add its conclusion to the *KB*, until query is found

$P \Rightarrow Q$

$L \wedge M \Rightarrow P$

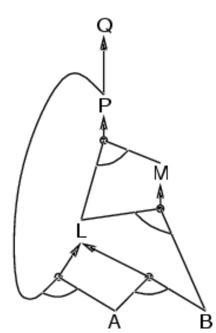
$B \wedge L \Rightarrow M$

$A \wedge P \Rightarrow L$

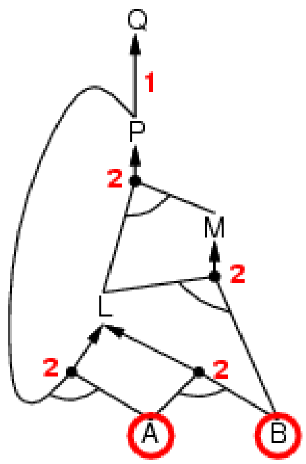
$A \wedge B \Rightarrow L$

A

B



Forward chaining example



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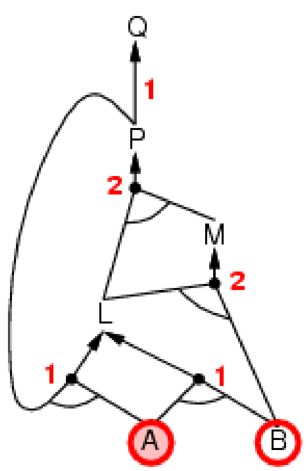

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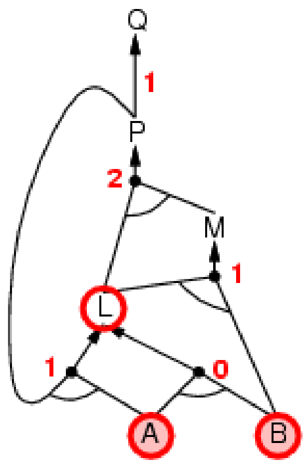

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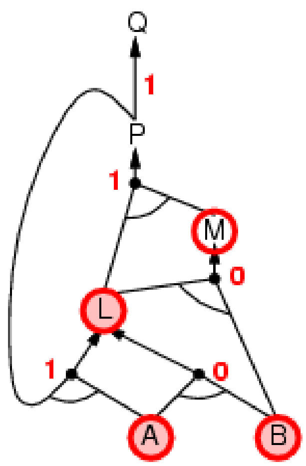
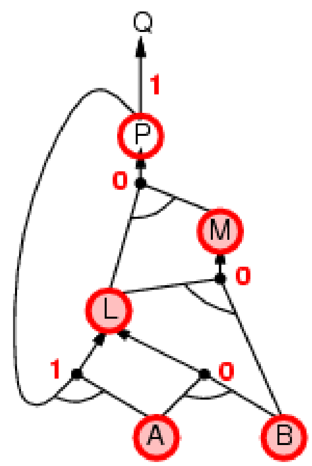
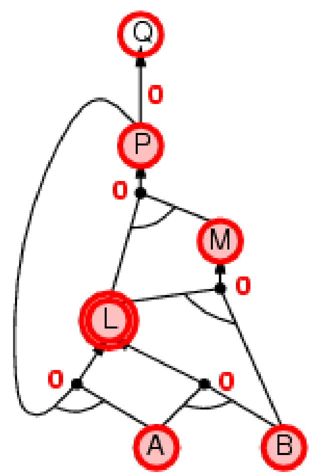


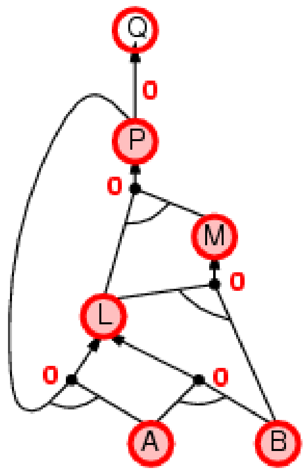
Forward chaining example



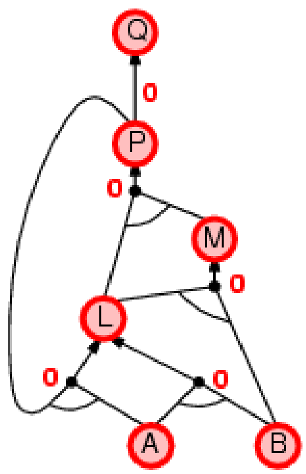
Forward chaining example



Forward chaining example



Forward chaining example



Backward chaining

Idea: work backwards from the query *q*:

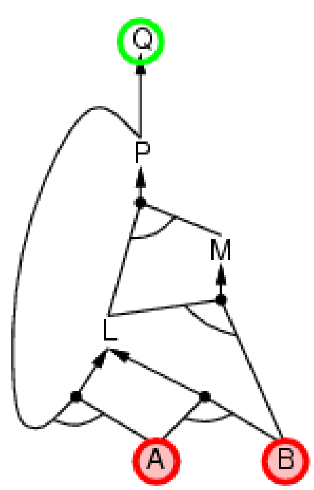
- to prove *q* by BC,
  - check if *q* is known already, or
  - prove by BC all premises of some rule concluding *q*

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1. has already been proved true, or
2. has already failed

Backward chaining example



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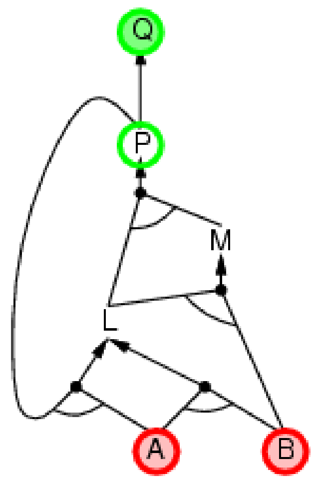

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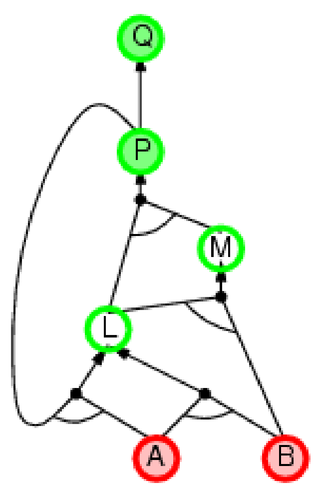

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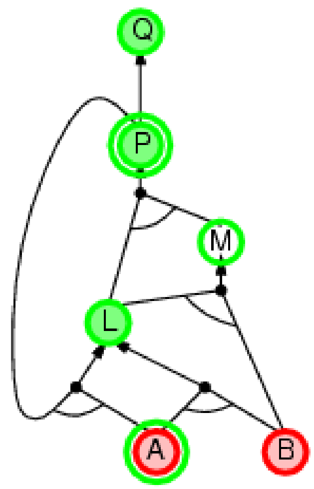

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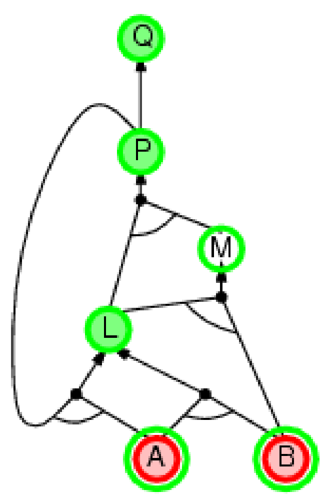

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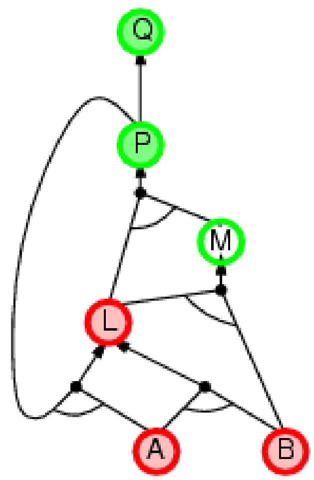

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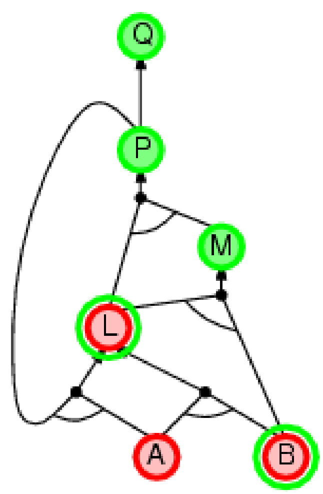

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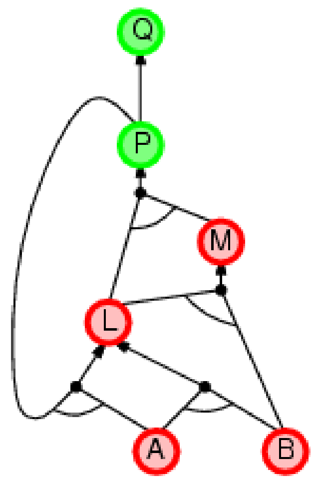

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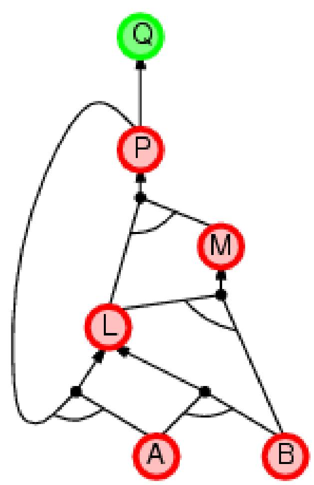

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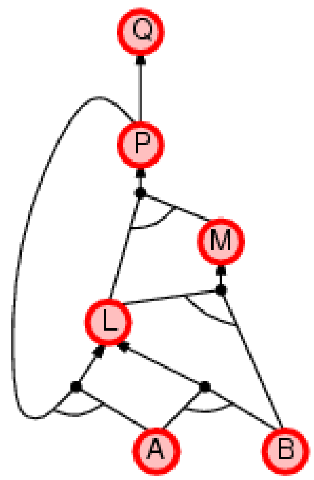

Forward vs. backward chaining

- FC is data-driven, automatic, unconscious processing,
  - e.g., object recognition, routine decisions
- May do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be much less than linear in size of KB

Efficient propositional inference

Two families of efficient algorithms for propositional inference:

Complete backtracking search algorithms

- DPLL algorithm (Davis, Putnam, Logemann, Loveland)
- Incomplete local search algorithms
  - `WalkSAT` algorithm

Expressiveness limitation of propositional logic

- KB contains "physics" sentences for every single square
- For every time *t* and every location [*x,y*],

$L^t_{x,y} \wedge FacingRight^t \wedge Forward^t \Rightarrow L^t_{x+1,y}$

- Rapid proliferation of clauses

Summary

- Logical agents apply inference to a knowledge base to derive new information and make decisions
- Basic concepts of logic:
  - syntax: formal structure of sentences
  - semantics: truth of sentences wrt models
  - entailment: necessary truth of one sentence given another
  - inference: deriving sentences from other sentences
  - soundness: derivations produce only entailed sentences
  - completeness: derivations can produce all entailed sentences
- Wumpus world requires the ability to represent partial and negated information, reason by cases, etc.
- Resolution is complete for propositional logic
  Forward, backward chaining are linear-time, complete for Horn clauses
- Propositional logic lacks expressive power