Constraint Satisfaction Problems

Chapter 5

Sections 1 – 3

Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

Constraint satisfaction problems (CSPs)

- Standard search problem:
  - state is a "black box" – any data structure that supports successor function, heuristic function, and goal test
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

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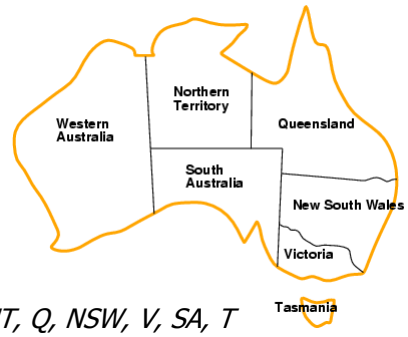
Example: Map-Coloring



- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}

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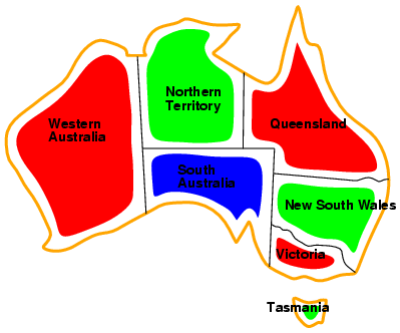
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

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Example: N-Queens

- Goal: n queens placed in non-attacking positions on the board
- Variables: ?
- Variable Values: ?
- Constraints: ?

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Example: Satisfiability (SAT)

- Determine whether a sentence in conjunctive normal form (CNF) is satisfiable (can evaluate to true)
  - Used in propositional logic (covered later)
  - (P V Q V ~R) ^ (~P V ~R V S) …
- Variables: ?
- Variable Values: ?
- Constraints: ?

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Example: Cryptograms

M GWD'X URNMRPR MD QD
I DON'T CEL I E E IN AN
QCXRLNMCR QNXFWZOF M QH
A FTERL I FE ALTHO GH I AM
ULMDOMDO Q KFQDOR WC
CR IN G I NG A HANGE OF
ZDGR LA RQL
NDE RWEAR

- Variables: ?
- Variable Values: ?
- Constraints: ?

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M GWD'X URNMRPR MD QD
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BRIN G I NG A CHANGE OF
ZDGRLARQL
UNDERWEAR

AWWGS QNNRD
WOODY ALLEN

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Varieties of CSPs

- Discrete variables
  - finite domains:
    - n variables, domain size $d \rightarrow O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

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Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables

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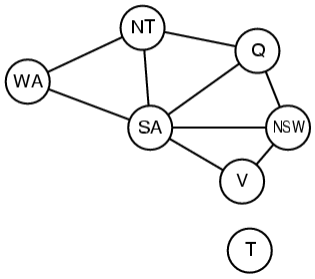
Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



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Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling

- Notice that many real-world problems involve real-valued variables

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Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it

States are defined by the values assigned so far

- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
  → fail if no legal assignments
- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   → use depth-first search
3. Path is irrelevant

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Backtracking search

- Variable assignments are commutative}, i.e.,

[WA = red then NT = green] same as [NT = green then WA = red]

- Only need to consider assignments to a single variable at each node
- Depth-first search for CSPs with single-variable assignments is called backtracking search
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve *n*-queens for $n \approx 25$

Backtracking search

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
    return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment according to Constraints[csp] then
            add { var = value } to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failue then return result
            remove { var = value } from assignment
    return failure
```

Backtracking example

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Backtracking example

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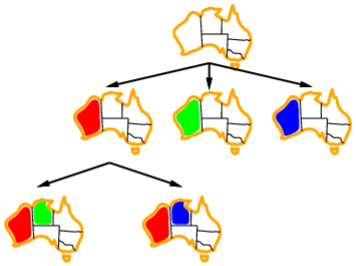
Backtracking example



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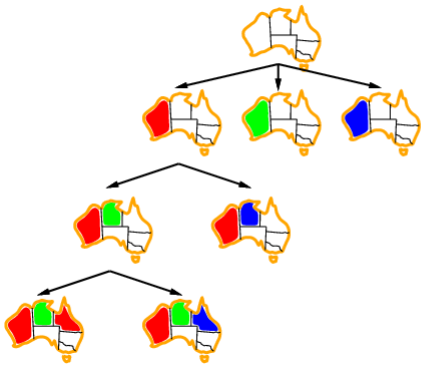
Backtracking example



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Solving 4-Queens CSP as Search

- Example Search Tree (in class)
- Maximum depth of the tree (m): ?
- Depth of the solution (d): ?
- Branching factor (b): ?

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Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

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Most constrained variable

- Most constrained variable:
  choose the variable with the fewest legal values

- a.k.a. minimum remaining values (MRV) heuristic

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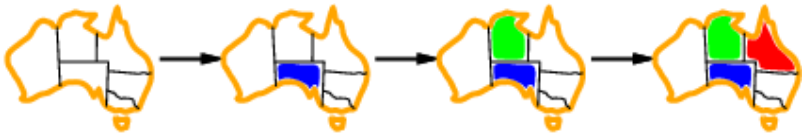
Most constraining variable

- Tie-breaker among most constrained variables
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables



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Least constraining value

- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables

- Combining these heuristics makes 1000 queens feasible

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Heuristics for CSP

- Another map coloring example (in class)
- Most constrained variable: ?
- Least constraining value: ?

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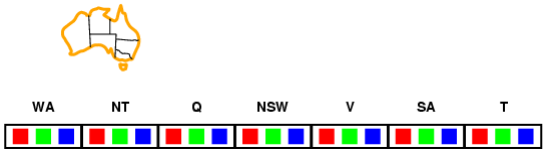
Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



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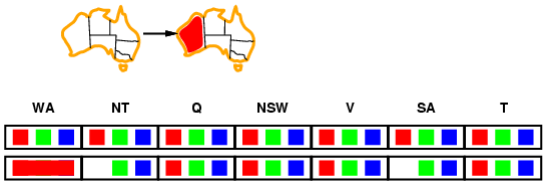

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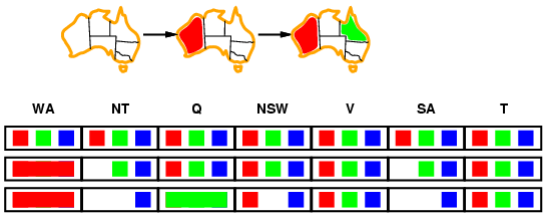

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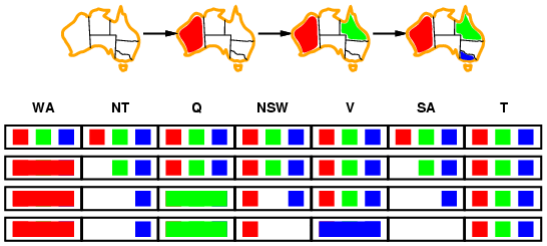

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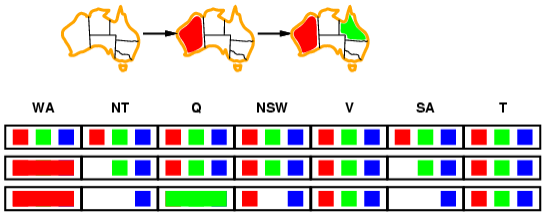
Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

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Constraint Propagation

- A state (more broadly) is defined by a set of variables, their values, and a list of legal and illegal assignments for unassigned variables
- Constraints + assignments can entail new legal and illegal assignments
- Constraint propagation: the process of infereinng new assignments from existing assignments

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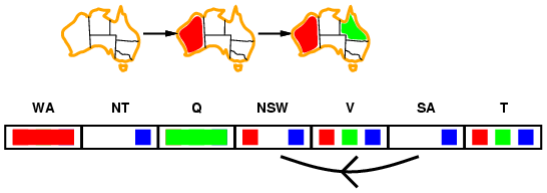
Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



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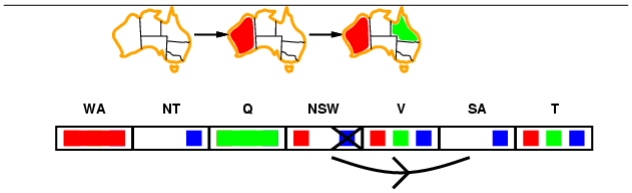

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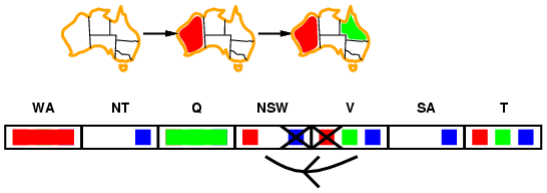

- If X loses a value, neighbors of X need to be rechecked

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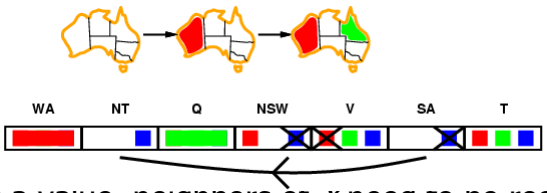

- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Constraint Propagation

- Another map-coloring example (in class)

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Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

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Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: *h(n)* = number of attacks

- Given random initial state, can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)

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Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies

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