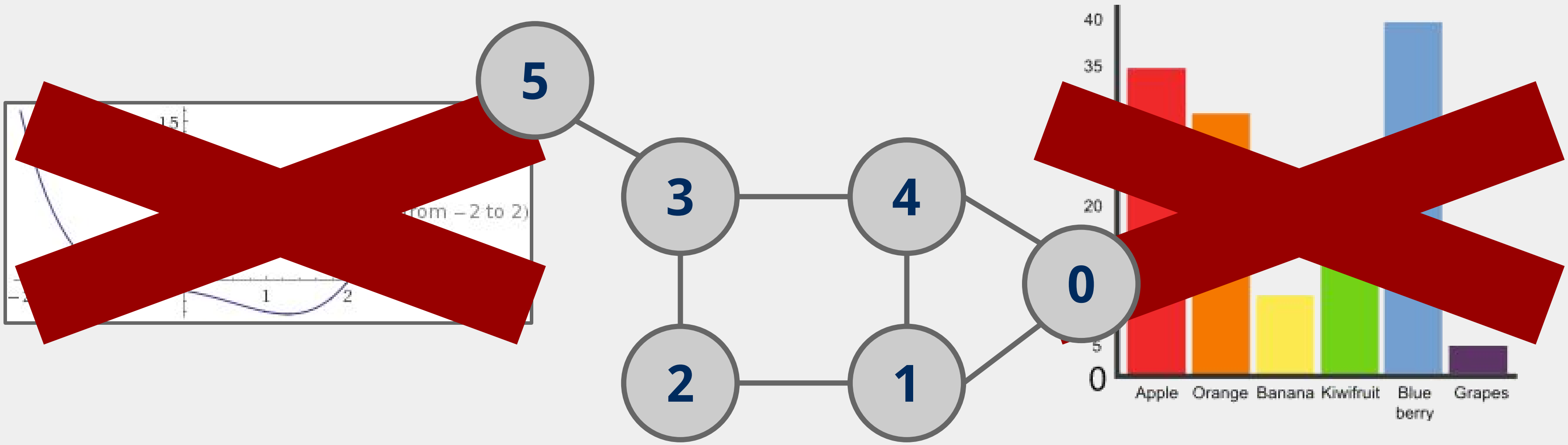
CS/COE 1501

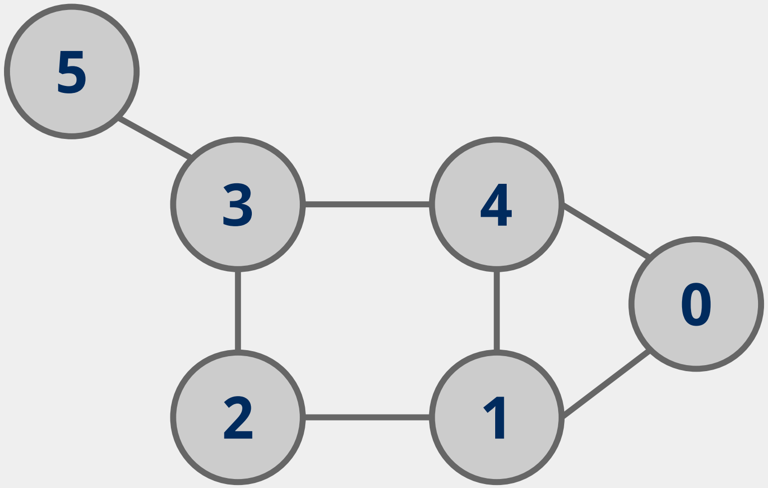
www.cs.pitt.edu/~lipschultz/cs1501/

Graphs


5
3
4
2
1
0
15
1
2
40
35
20
5
0
Apple
Orange
Banana
Kiwifruit
Blue berry
Grapes

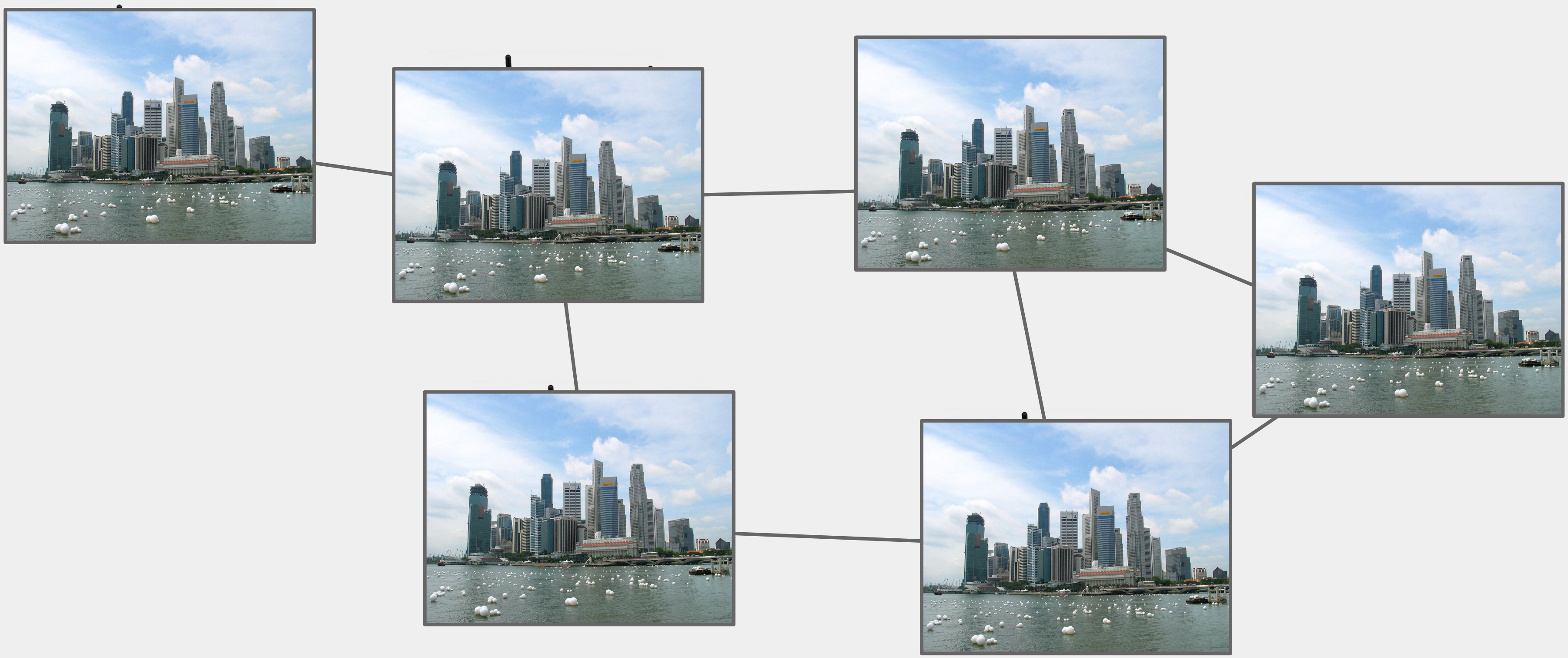
Graphs

- A graph G = (V, E)
  - Where V is a set of vertices
  - E is a set of edges connecting vertex pairs
- Example:
  - V = {0, 1, 2, 3, 4, 5}
  - E = {(0, 1), (0, 4), (1, 2), (1, 4), (2, 3), (3, 4), (3, 5)}



Why?

- Can be used to model many different scenarios



Some definitions

- Undirected graph
  - Edges are unordered pairs: (A, B) == (B, A)
- Directed graph
  - Edges are ordered pairs: (A, B) != (B, A)
- Adjacent vertices, also called “neighbors”
  - Vertices connected by an edge

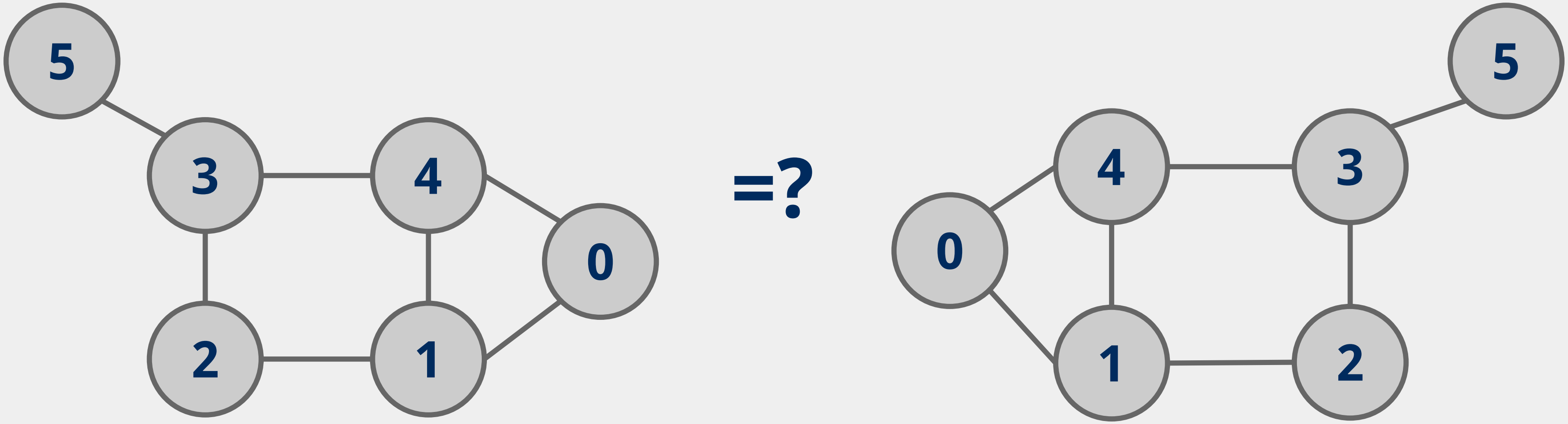
Graph sizes

- Let v = |V|, and e = |E|
- Given v, what are the minimum/maximum sizes of e?
  - Minimum value of e?
    - Definition doesn't necessitate that there are any edges…
    - So, 0
  - Maximum of e?
    - Are self edges allowed?
    - Directed graph? (We'll assume self edges)
      - v vertices each with edges to v vertices
      - v^2
    - Undirected graph? (We'll assume NO self edges)
      - v vertices with edges to v-1 vertices
        - BUT remember for undirected, (1, 2) == (2, 1)
      - v(v - 1) / 2
        - $(v^2 - v) / 2$

More definitions

- A graph is considered *sparse* if:
  - e <= v lg v
- A graph is considered *dense* as it approaches the maximum number of edges
  - $e = v^2 - \varepsilon$ for directed
  - $e = ((v^2 - v) / 2) - \varepsilon$ for undirected
- A *complete* graph has the maximum number of edges

Representing graphs



- Typically, yes. Different spatial representations of the same vertices/edges are considered the same graph.
- Trivially, graphs can be represented as:
  - List of vertices
  - List of edges
  - What operations would we want to perform on these lists?

Using an adjacency matrix

- Rows/columns are vertex labels
  - M[i][j] = 1 if $(i, j) \in E$
  - M[i][j] = 0 if $(i, j) \notin E$

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	1	0	1	0	1	0
2	0	1	0	1	0	0
3	0	0	1	0	1	1
4	1	1	0	1	0	0
5	0	0	0	1	0	0

Adjacency matrix analysis

- Pros?
  - Easy to use/intuitive
  - Runtime for checking edge existence?

- Cons?
  - Memory
  - Time to initialize
  - Time to find neighbors of a vertex

Adjacency lists

- Array of neighbor lists
  - A[i] contains neighbors of vertex i

- See example

- Pros?
  - Memory
  - Time to find the neighbors of a node
- Cons?
  - Memory
  - Time to check edge existence

In general…

- Adjacency matrix is better for dense graphs
- Adjacency list is better for sparse graphs

Even more definitions

- Path
  - A sequence of adjacent vertices
- Simple Path
  - A path in which no vertices are repeated
- Simple Cycle
  - A simple path with the same first and last vertex
- Connected Graph
  - A graph in which a path exists between all vertex pairs
- Connected Component
  - Connected subgraph of a graph
- Acyclic Graph
  - A graph with no cycles
- Tree
  - ?
  - A connected, acyclic graph
    - Has exactly v-1 edges

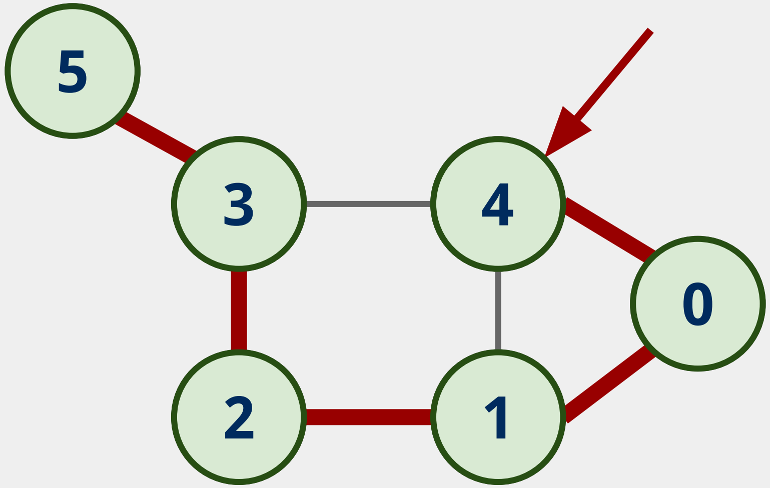
Graph traversal

- What is the best order to traverse a graph?
- Two primary approaches:
  - Depth-first search (DFS)
    - "Dive" as deep as possible into the graph first
    - Branch when necessary
  - Breadth-first search (BFS)
    - Search all directions evenly
      - I.e., from i, visit all of i's neighbors, then all of their neighbors, etc.

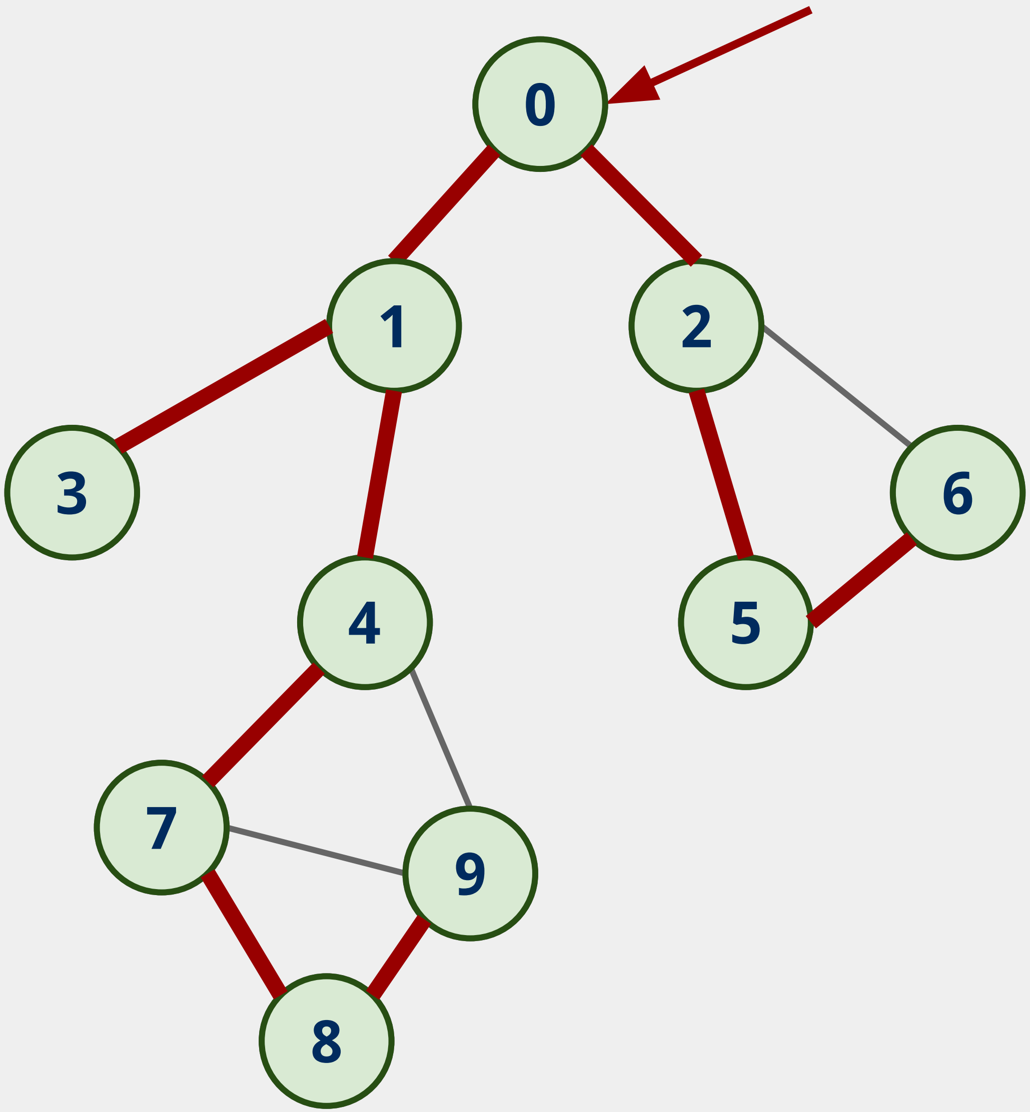
DFS

- Already seen and used this throughout the term
  - For tries…
  - For Huffman encoding…
- Can be easily implemented recursively
  - For each node, visit first unseen neighbor
  - Backtrack at dead ends (i.e., nodes with no unseen neighbors)
    - Try next unseen neighbor after backtracking

DFS example



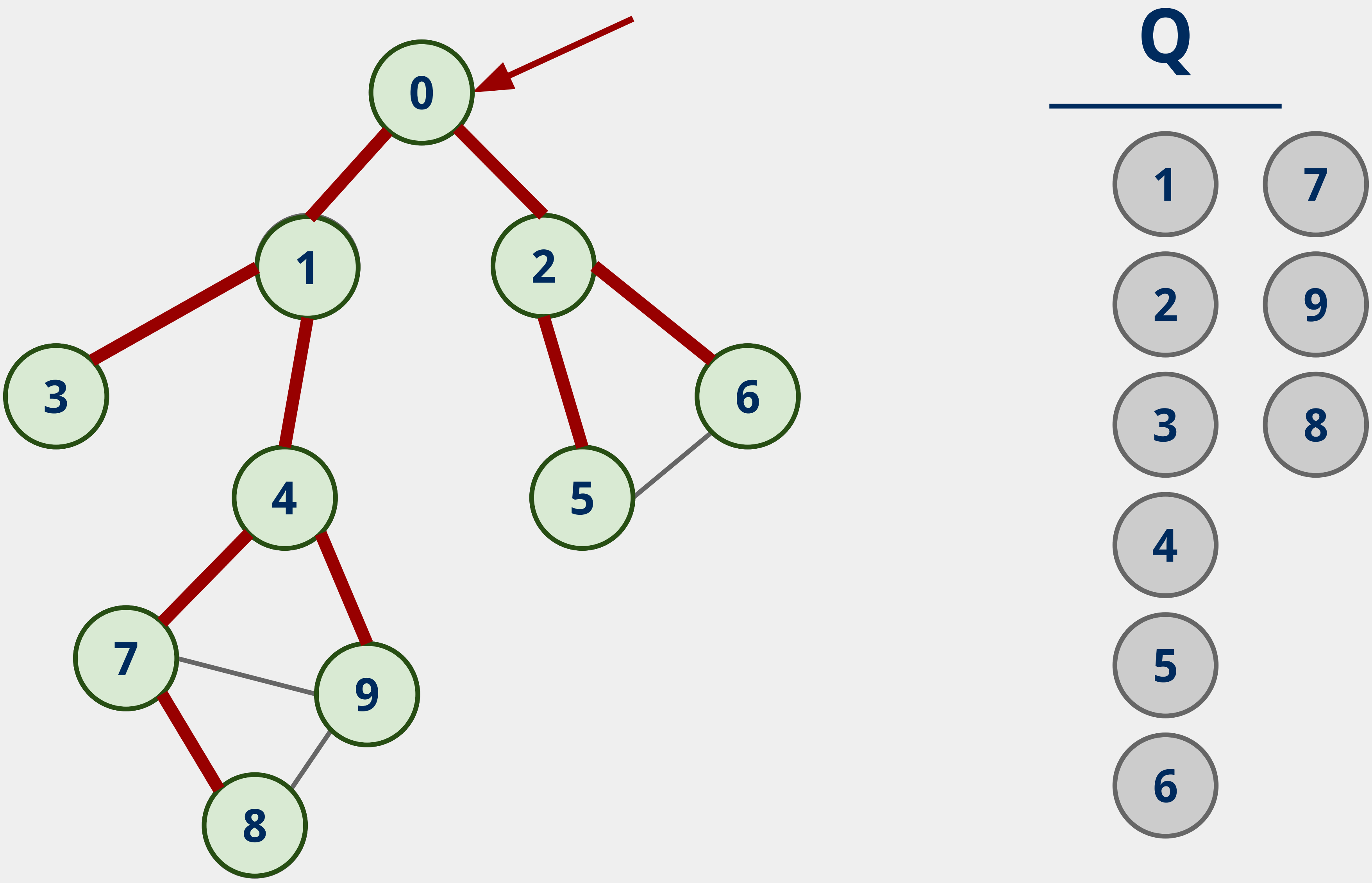
DFS example 2



BFS

- Can be easily implemented using a queue
  - For each node visited, add all of its neighbors to the queue
    - Vertices that have been seen but not yet visited are said to be the *fringe*
  - Pop head of the queue to be the next visited vertex
- See example

BFS example



DFS and BFS would be called from a wrapper function

- If the graph is connected:
  - dfs()/bfs() is called only once and returns a *spanning tree*
- Else:
  - A loop in the wrapper function will have to continually call dfs()/bfs() while there are still unseen vertices
  - Each call will yield a spanning tree for a connected component of the graph

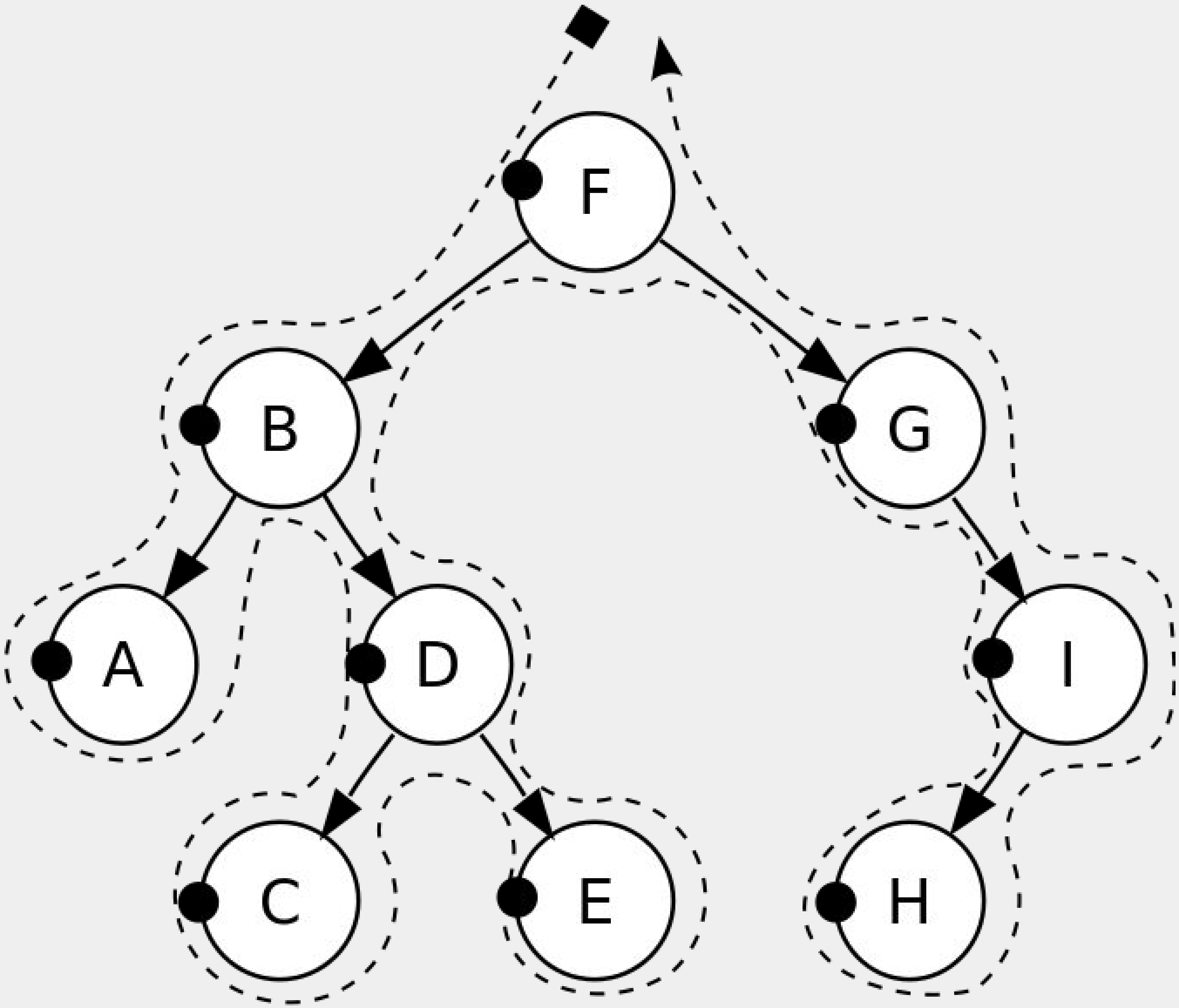
Shortest paths

- BFS traversals can further be used to determine the *shortest path* between two vertices

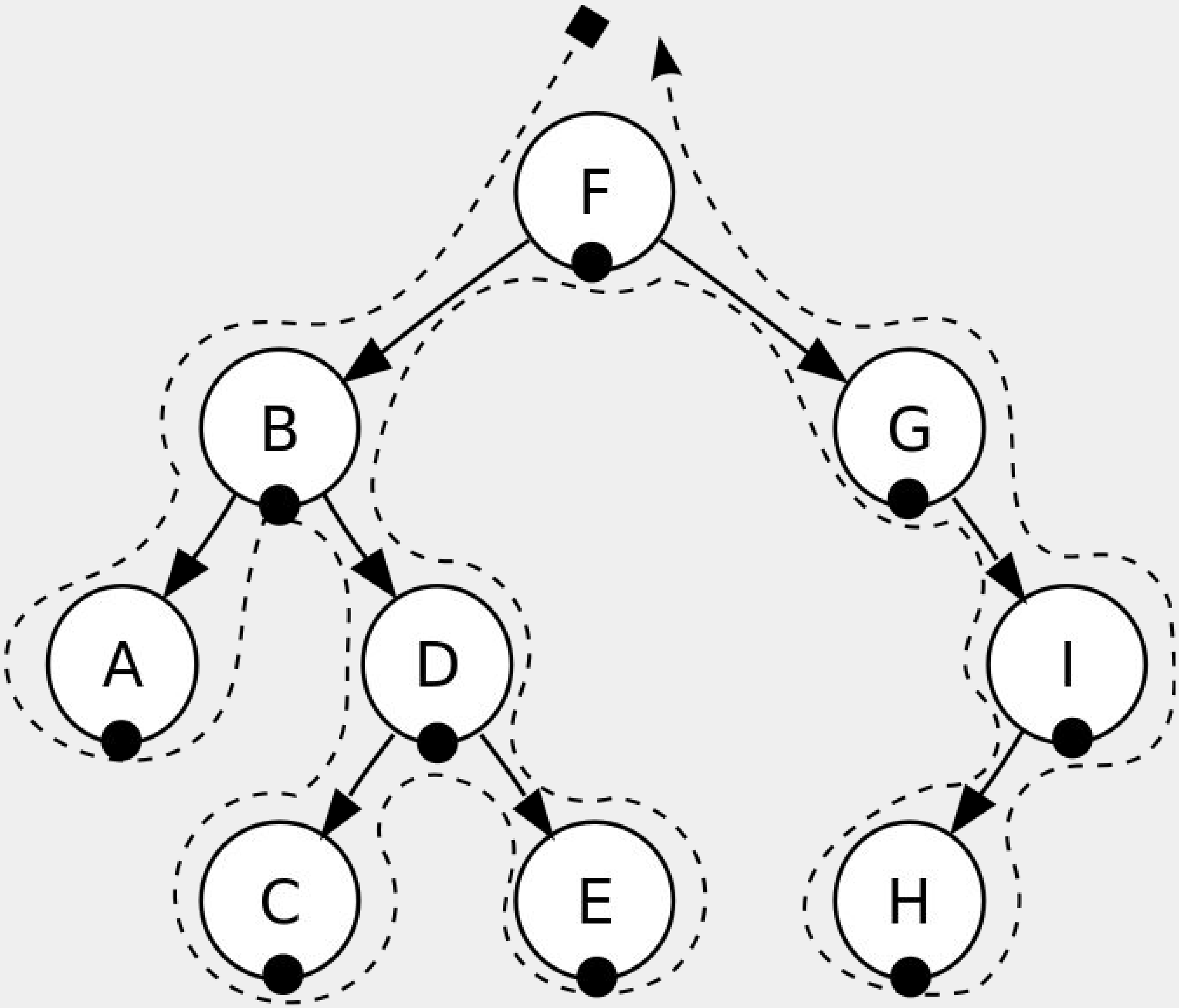
Analysis of graph traversals

- At a high level, DFS and BFS have the same runtime
  - Each node must be seen and then visited, but the order will differ between the two approaches
- Adjacency matrix
  - Θ(v) to consider all neighbors
    - To traverse a row/column of the matrix
    - Doing this for each of v vertices leads to $\Theta(v^2)$ runtime
- Adjacency list
  - Must consider the neighbor list of each node in the array
    - So, we must visit every node in the whole adjacency list
      - Θ(v + e)
        - Why not just e?

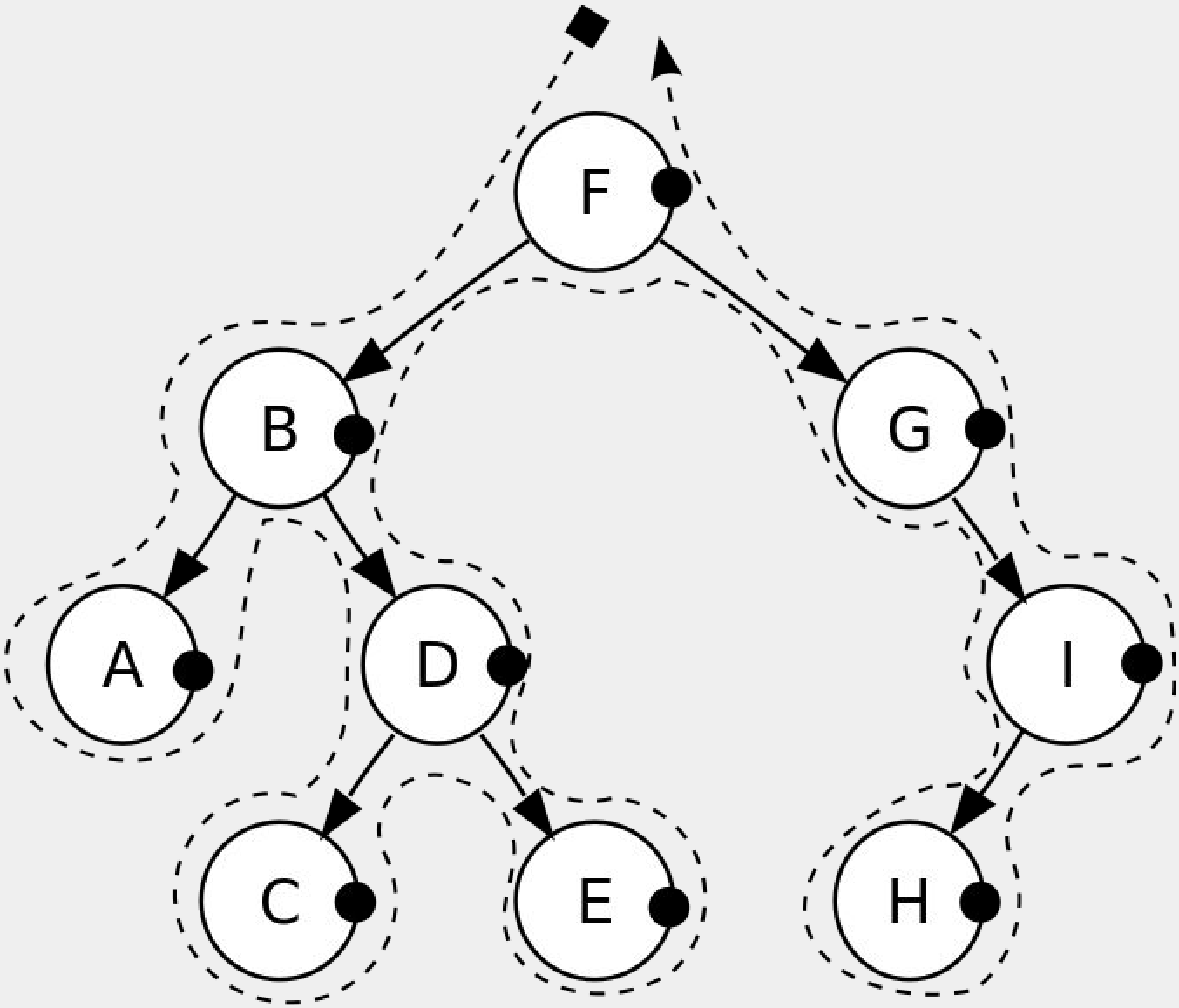
DFS pre-order traversal



DFS in-order traversal



DFS post-order traversal



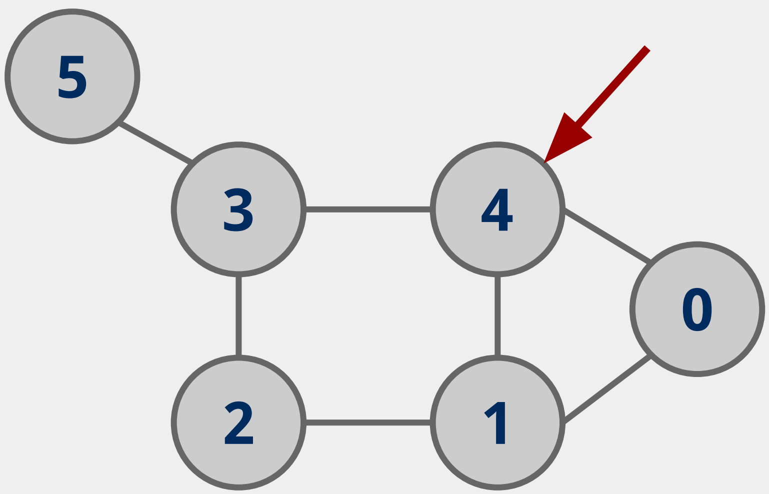
Biconnected graphs

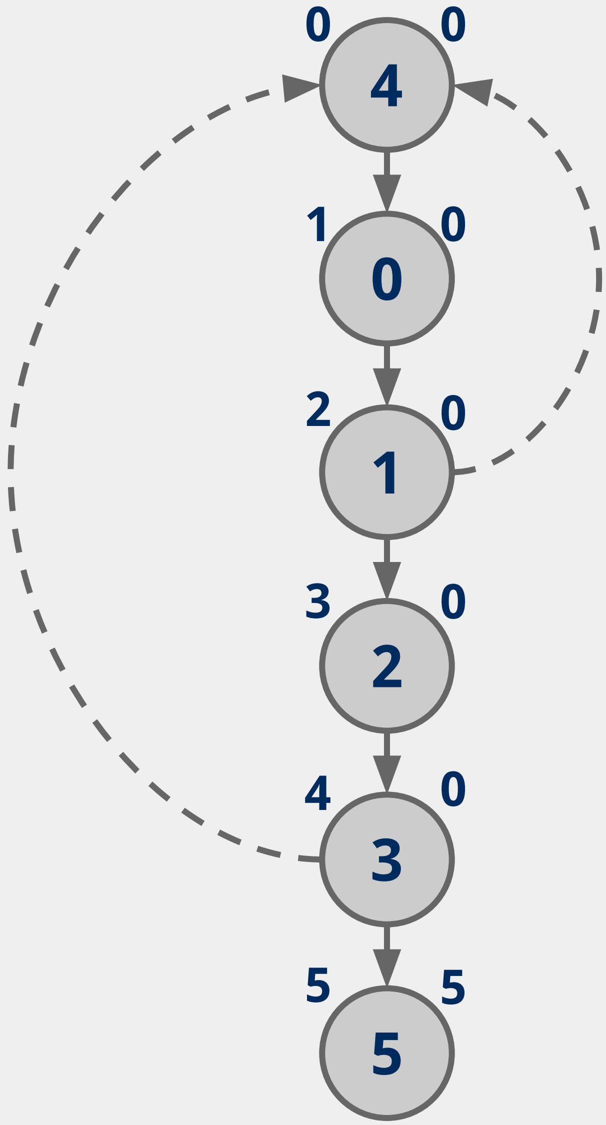
- A *biconnected graph* has at least 2 distinct paths (no common edges or vertices) between all vertex pairs
- Any graph that is not biconnected has one or more *articulation points*
  - Vertices, that, if removed, will separate the graph
- Any graph that has no articulation points is biconnected
  - Thus we can determine that a graph is biconnected if we look for, but do not find, any articulation points

Finding articulation points

- Variation on DFS
- Consider building up the spanning tree
  - Have it be directed
  - Create "back edges" when considering a node that has already been visited in constructing the spanning tree
  - Label each vertex v with with two numbers:
    - num(v) = pre-order traversal order
    - low(v) = lowest-numbered vertex reachable from v using 0 or more spanning tree edges and then at most one back edge
      - Min of:
        - num(v)
        - Lowest num(w) of all back edges (v, w)
        - Lowest low(w) of all spanning tree edges (v, w)

Finding articulation points example





So where are the articulation points?

- If any (non-root) vertex v has some child w such that low(w) ≥ num(v), v is an articulation point
- What about if we start at an articulation point?
    - If the root of the spanning tree has more than one child, it is an articulation point