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CS 3150 Randomness and Computation
Homework 8.8 **Due:** Wednesday, March 7, 2006

Chapter 8, Exercise 8.8 (page 219)

At the first step of the algorithm, we consider the $n-1$ edges from vertex 1 with i.i.d. uniformly distributed weights and select their minimum with expected value $\frac{1}{n}$ [Lemma 8.3; page 195]. At step 2, we consider $n-2$ edges; those except the edge to vertex 1. In general, at step i, we select the minimum of $n-i$ i.i.d. uniformly distributed edge weights with expected value $\frac{1}{n-i+1}$. The last edge (that completes the cycle) has an expected value of $\frac{1}{2}$. Thus, the expected weight of the Hamiltonian cycle is:

$\mathbf{E}[\text{cycle weight}] = \sum_{i=1}^{n-1} \mathbf{E}[\text{weight of edge added at step } i] + \mathbf{E}[\text{last edge}] = \frac{1}{2} + \sum_{i=1}^{n-1} \frac{1}{n-i+1} = ln(n) + O(1)$

Then, we consider When edge weights are i.i.d. exponential random variables with parameter 1. At step i, we select the minimum of $n-i$ i.i.d. exponentially distributed edge weights with expected value $\frac{1}{n-i}$ [Lemma 8.5.; Page 198]. The expected weight of the last edge is 1.

$\mathbf{E}[\text{cycle weight}] = \sum_{i=1}^{n-1} \mathbf{E}[\text{weight of edge added at step } i] + \mathbf{E}[\text{last edge}] = 1 + \sum_{i=1}^{n-1} \frac{1}{n-i} = ln(n-1) + O(1)$

Chapter 8, Exercise 8.10 (page 220)

(a) Each arc starts at a point and its length is measured in some direction, either clockwise or anti-clockwise. We first find the probability that an arc with length Z_i is longer than $c\frac{\ln n}{n-1}$. This happens when there is a gap of length greater than $c\frac{\ln n}{n-1}$, that is all the other $n-1$ points do not fall into the gap. $Pr\{Z_i > c\frac{\ln n}{n-1}\} = (1 - c\frac{\ln n}{n-1})^{n-1} \sim e^{-c \ln n} = \frac{1}{n^c}$. Using the Union Bound, the probability that one or more $Z_i > c\frac{\ln n}{n-1}$ is $\leq n \cdot \frac{1}{n^c} = \frac{1}{n^{c-1}}$, because we have n arcs each with probability $\frac{1}{n^c}$ of exceeding $c\frac{\ln n}{n-1}$ in length. The probability that all Z_i are less than $c\frac{\ln n}{n-1}$ is thus one minus the above probability, that is, $\geq 1 - \frac{1}{n^{c-1}}$.

(b) Define the 0/1 random variable $X_i = 1$ if $Z_i \geq c' \ln n$, and $X_i = 0$ otherwise. Define $X = \sum_{i=1}^{n} X_i$. From Thm 6.10 (page 137), $\Pr\{X > 0\} \geq \sum_{i=1}^{n} \frac{Pr(X_i=1)}{\mathbf{E}[X|X_i=1]}$

(c) Similar to part (a), $Pr\{Z_i < \frac{1}{2n^2}\} \leq (n-1) \cdot \frac{1}{2n^2} \leq \frac{1}{2n}$. $\Pr\{\text{one or more } Z_i < \frac{1}{2n^2}\} \leq n \cdot \frac{1}{2n} = \frac{1}{2}$. And, finally, $\Pr\{\text{all } Z_i \geq \frac{1}{2n^2}\} = 1 - \Pr\{\text{one or more } Z_i < \frac{1}{2n^2}\} \geq \frac{1}{2}$.