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CS 3150 Randomness and Computation
Homework 7.18 **Due:** Wednesday, Feb 23, 2006

Chapter 7, Exercise 7.18 (page 184)

(a) Consider a random walk on the 2-dimensional integer lattice, where each point has four neighbors (up, down, left, and right). Is each state transient, null recurrent, or positive recurrent? Give an argument.

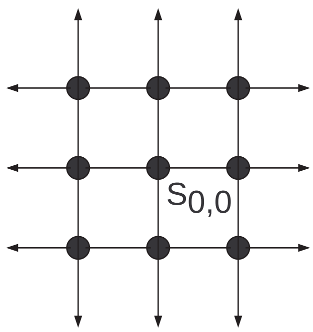

Figure 1: A 2-D integer lattice

Figure 1 shows the infinite integer lattice on which we define each transition probability to be 1/4. As we can see, it is possible to go out a distance, and then return back to the origin ($S_{0,0}$). In some sense, there is a 3/4 probability of moving away from the origin at any step, and a 1/4 chance of making a beneficial step. Using this loose interpretation, we can use a modified representation that is easier to reason about, which we show in Figure 2.

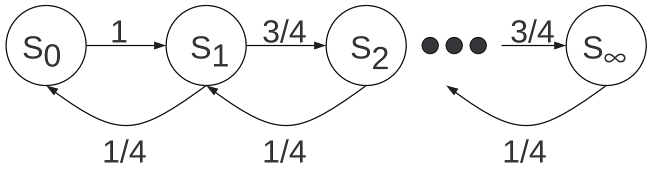

Figure 2: A chain representation of the lattice

The chain fails to handle the case that you have moved to a state where there are multiple paths that can return you to $S_{0,0}$ in the same time (meaning that the probability of moving to a "better" state is greater than 1/4.) For instance, being in state $S_{1,1}$ affords two paths of length 2 back to $S_{0,0}$ giving a proper transition probability of 1/2. It stands to reason that it is more difficult to get back to S_0 in the chain of Figure 2, so if we can prove that this chain is recurrent, we have proven it for the lattice.

Following the procedure outlined under Definition 7.5 (page 165), to show that the lattice is recurrent and not transient, we need to show that the probability of never returning to S_0 is zero. The probability of never returning can be thought of as the probability of moving away from the origin each step. Notice this doesn't take into account loops where we come back and then go out again, but asymptotically we will always be moving away. An equation to represent this is:

$$\prod_{j=1}^{t} \frac{3}{4} = \left(\frac{3}{4}\right)^{t}$$

Which approaches zero as t grows infinitely large.

Knowing that we must eventually return, the next question is whether we return in finite time. We can model the probability of returning at time t as:

$$r_{00}^{t} = \left(\frac{3}{4}\right)^{t/2-1} \left(\frac{1}{4}\right)^{t/2} \tag{1}$$

We use the definition of hitting time as below:

$$\begin{aligned} h_{00} &= \sum_{t=1}^{\infty} t \cdot r_{00}^{t} \\ &= \sum_{t=1}^{\infty} t \left(\frac{3}{4}\right)^{t/2-1} \left(\frac{1}{4}\right)^{t/2} \\ &= \frac{16}{\sqrt{3}\left(-4+\sqrt{3}\right)^{2}} \end{aligned} \tag{2}$$

Since this converges and is not unbounded, the chain must be *positive recurrent*. We can apply the same reasoning as before to say that the integer lattice is positive recurrent since our chain makes it more difficult to return to the origin. Notice however that the exact value does not hold for the lattice.

Being positive recurrent is surprising since the necessary requirement for a null- recurrent state is an infinite set of states, which we have. A possible intuition is that due to the relatively high probabilities, and the high connectivity, that a random walk never travels too far from the origin.

(b) Answer the problem in (a) for the 3-dimensional integer lattice.

Using the solution from part (a), we can make similar inferences about a 3-dimensional lattice. We modify Equation 1 to use the probabilities 1/6 and 5/6, and plug into Equation 2. This yields the finite result:

$$h_{00} = \frac{36\sqrt{5}}{\left(-6+\sqrt{5}\right)^2}$$

Which means the 3-dimensional integer lattice is also positive-recurrent.