

Chapter 6, Exercise 6.8 (page 150)

Prove that, for every integer n , there exists a way to 2-color the edges of K_n so that there is no monochromatic clique of size k when

$$x = n - \binom{n}{k} 2^{1-\binom{k}{2}}.$$

(Hint: Start by 2-coloring the edges of K_n , then fix things up.)

Consider a 2-coloring of K_n chosen uniformly at random out of the $2^{\binom{n}{2}}$ possible 2-colorings. Fix an arbitrary ordering of the $\binom{n}{k}$ k -cliques in K_n , and for $i = 1, \dots, \binom{n}{k}$ let X_i be an indicator r.v. such that $X_i = 1$ if clique i is monochromatic and 0 otherwise. Further let $X = \sum_{i=1}^{\binom{n}{k}} X_i$. Then, by linearity of expectations, we have

$$\begin{aligned} \mathbf{E}[X] &= \sum_{i=1}^{\binom{n}{k}} \mathbf{E}[X_i] \\ &= \binom{n}{k} 2^{-\binom{k}{2}+1}. \end{aligned}$$

Suppose we remove a vertex from each monochromatic k -clique. Then the residual graph is a complete graph without monochromatic k -cliques. The expected number of vertices in the residual graph is

$$\begin{aligned} n - \mathbf{E}[X] &= n - \binom{n}{k} 2^{1-\binom{k}{2}} \\ &= x. \end{aligned}$$

Then, by Lemma 6.2, there is a 2-coloring of K_n where removing a vertex from each monochromatic k -clique results in a K_y , $y \geq x$ that does not contain any monochromatic k -cliques. Observe that for any $y > x$, K_y has $\binom{y}{x}$ K_x subgraphs. If K_y does not have a monochromatic K_k then none of $\binom{y}{x}$ K_x subgraphs has one.