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CS 3150 Randomness and Computation
Homework 7 **Due: Wednesday, February 8, 2006**

Chapter 6, Exercise 6.14 (page 151)

For the graph $G_{n,p}$ where $p = 1/n$, let X be the number of clique triangles, show that $\Pr(X \geq 1) \leq 1/6$ and $lim_{n\to\infty} Pr(X \geq 1) \geq 1/7$.

For the first inequality, the number of ways to choose a triangle from the graph is $\binom{n}{3}$. The probability of each triangle is $p^3 = \frac{1}{n^3}$. Thus,

$$\begin{aligned} Pr(X \geq 1) &= \binom{n}{3} \frac{1}{n^3} \\ &= \frac{n(n-1)(n-2)}{6n^3} \leq 1/6. \end{aligned}$$

For the second inequality, we use the conditional expectation inequality as follows:

$$\begin{aligned} Pr(X \geq 1) &= \sum_{i=1}^{\binom{n}{3}} \frac{Pr(X_i = 1)}{\mathbf{E}[X|X_i = 1]} \\ &= \sum_{i=1}^{\binom{n}{3}} \frac{p^3}{1 + \binom{n-3}{3}p^3 + 3\binom{n-4}{2}p^2 + 3\binom{n-5}{1}p} \\ &= \frac{\binom{n}{3}p^3}{1 + \binom{n-3}{3}p^3 + 3\binom{n-4}{2}p^2 + 3\binom{n-5}{1}p} \\ \lim_{n\to\infty} Pr(X \geq 1) &= 1/6 \geq 1/7. \end{aligned}$$