

Random Algorithm HW1

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Let X_k be the number of white balls in the bin while there are k balls in the bin, T_k be the random event of pick when there are k balls in the bin.

We want to prove that for any $1 \leq i, j \leq k-1$, $Pr(X_k = i) = Pr(X_k = j)$.

We can prove this by induction:

$k = 2$, then it is obvious that $i = j = 1$, $Pr(X_k = i) = Pr(X_k = j)$ holds.

Now assume that when $k = m$, for any $1 \leq i, j \leq m-1$, $Pr(X_m = i) = Pr(X_m = j)$.

then for $k = m+1$, for any $1 \leq i, j \leq m$,

$$Pr(X_{m+1} = i) = Pr(X_m = i-1, T_m = White) + Pr(X_m = i, T_m = Black)$$

if $1 < i < m$

$$= Pr(T_m = White|X_m = i-1) * Pr(X_m = i-1) + Pr(T_m = Black|X_m = i) * Pr(X_m = i)$$

$$= (Pr(T_m = White|X_m = i-1) + Pr(T_m = Black|X_m = i)) * Pr(X_m = j)$$

$$= \left(\frac{i-1}{m} + \frac{m-i}{m}\right) * Pr(X_m = j)$$

if $i = m$

$$= Pr(T_m = White|X_m = m-1) * Pr(X_m = m-1) + Pr(T_m = Black|X_m = m) * Pr(X_m = m)$$

$$= Pr(T_m = White|X_m = m-1) * Pr(X_m = j) \text{ because } (Pr(X_m = m) = 0)$$

if $i = 1$

$$= Pr(T_m = White|X_m = 0) * Pr(X_m = 0) + Pr(T_m = Black|X_m = 1) * Pr(X_m = 1)$$

$$= Pr(T_m = Black|X_m = 1) * Pr(X_m = j) \text{ because } (Pr(X_m = 0) = 0)$$

in all cases

$$= \frac{m-1}{m} * Pr(X_m = j)$$

So the proof is done.