CS 3150
Homework Assignment Chapter 5

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1 Chapter 5, Exercise 5.11

The following problem models a simple distributed system wherein agents contend for resources but "backoff" in the face of contention. Balls represent agents and bins represent resources.

The system evolves over rounds. Every round, balls are thrown independently and uniformly at random into n bins. Any ball that lands in a bin by itself is served and removed from consideration. The remaining balls are thrown again in the next round. We begin with n balls in the first round, and we finish when every ball is served.

a *If there are b balls at the start of a round, what is the expected number of balls at the start of the next round?*
Let X_i be a binary variable indicating the event that bin i has only one ball. For a given bin:

$$\mathbf{Pr}[X_i = 1] = \lim_{n \to \inf} \frac{e^{-\frac{b}{n}}(\frac{b}{n})}{1!} = \frac{b}{n}e^{-\frac{b}{n}}$$

Where the distribution of balls is assumed to be Poisson with a mean of m/n

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} \mathbf{E}[X_i]\right] = be^{-\frac{b}{n}}$$

Since balls that fall in a bin only by themselves are removed, the number of balls at the start of the next round will be:

$$\text{Balls remaining} = b(1 - e^{-\frac{b}{n}}) \tag{1}$$

b *Suppose that every round the number of balls served was exactly the expected number of balls to be served. Show that all the balls would be served in O(loglogn) rounds.*

Using $x_{j+1} \leq {x_j}^2/n$ and (1):

$$x_{j+1} \leq \frac{x_j^2}{n} \tag{2}$$

$$x_j(1 - e^{-\frac{x_j}{n}}) \leq \frac{x_j^2}{n}$$
$$(1 - e^{-\frac{x_j}{n}}) \leq \frac{x_j}{n}$$

Let $k = x_j/n$

$$1 - e^{-k} \leq k \qquad \text{true for } k \geq 0$$

Let x_0 be the initial number of balls. Then using (2):

$$
\begin{aligned}
x_1 &\leq \frac{{x_0}^2}{n} \\
x_2 &\leq \frac{(\frac{{x_0}^2}{n})^2}{n} = \frac{{x_0}^4}{n^3} \\
x_3 &\leq \frac{(\frac{{x_0}^4}{n^3})^2}{n} = \frac{{x_0}^8}{n^7} \\
&\quad . \\
&\quad . \\
x_j &\leq \frac{{x_0}^{2^j}}{n^{2^j - 1}}
\end{aligned}
$$

Let $x_{k-1} = 1$, then all balls will be served in k rounds.

$$
\begin{aligned}
n^{2^{k-1}-1} &\leq {x_0}^{2^{k-1}} \\
(2^{k-1} - 1) logn &\leq 2^{k-1} log x_0 \\
2^{k-1}(logn - logx_0) - logn &\leq 0 \\
2^{k-1} &\leq \frac{logn}{logn - logx_0} \\
k - 1 &\leq loglogn - log(logn - logx_0) \\
\text{Since } x_0 = n & \\
k - 1 &\leq loglogn \\
\Rightarrow k &= O(loglogn)
\end{aligned}
$$

2 Chapter 5, Exercise 5.18

An undirected graph on n vertices is disconnected if there exists a set of $k < n$ vertices such that there is no edge between this set and the rest of the graph. Otherwise the graph is said to be connected. Show that there exists a constant c such that if $N \geq cnlogn$ then, with probability $O(e^{-n})$, a graph randomly chosen from $G_{n,N}$ is connected.
We can model this problem in the context of ball and bins by considering n bins and mean $\lambda = cnlogn/(n/2)$ balls in every bin.

$$\Rightarrow \qquad \lambda = 2clogn$$

$$\begin{aligned} \mathbf{Pr}[\text{a single node is disconnected}] &= e^{-2clogn} \\ &= (e^{-logn})^{2c} \\ &= \frac{1}{n^{2c}} \end{aligned}$$

$$\begin{aligned} \mathbf{Pr}[\text{no single node is disconnected}] &= (1 - \frac{1}{n^{2c}})^n \\ &= e^{-\frac{n}{n^{2c}}} \\ &= e^{-n^{1+2c}} \end{aligned}$$

$$\mathbf{Pr}[\text{graph is connected}] = \bigcap_{k=1}^{n/2} \mathbf{Pr}[\text{no k nodes are disconnected}]$$

$$\Rightarrow \mathbf{Pr}[\text{graph is connected}] \leq e^{-n^{1+2c}}$$

which is $O(e^{-n})$ for $c = 0$