

LP-based Approximation Algorithms for Broadcast Scheduling

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Class Presentation

CS3150: Approximation Algorithms

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B. Kalyanasundaram, K. Pruhs, and M. Velauthapillai

Scheduling broadcasts in wireless networks. Proc. *ESA'00*.

R. Gandhi, S. Khuller, Y.-A. Kim, and Y.-C. Wan

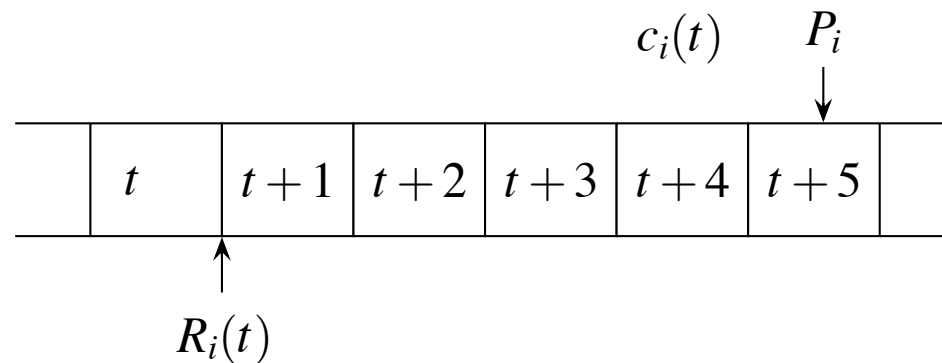
Algorithms for min. response time in broadcast scheduling. Proc. *IPCO'02*.

R. Gandhi, S. Khuller, S. Parthasarathy, and A. Srinivasan

Dependent rounding in bipartite graphs. Proc. *FOCS'02*

Offline Broadcast Scheduling [ESA00]

- Broadcast server with slotted broadcast channel
- n one-slot pages P_1, P_2, \dots, P_n
- Requests $R_i(t)$, $1 \leq i \leq n$ are received at the end of slot t
- Objective is to minimize avg. (total) response time



Resource Augmentation

- Offline problem shown NP-hard
- Ultimate goal: $O(1)$ approximation algorithm
- Resource augmentation [ESA00]: Can we get an $O(1)$ approximation alg. by throwing $O(1)$ more bandwidth at the problem?
- A k -speed server is one that broadcasts up to k pages every time slot.

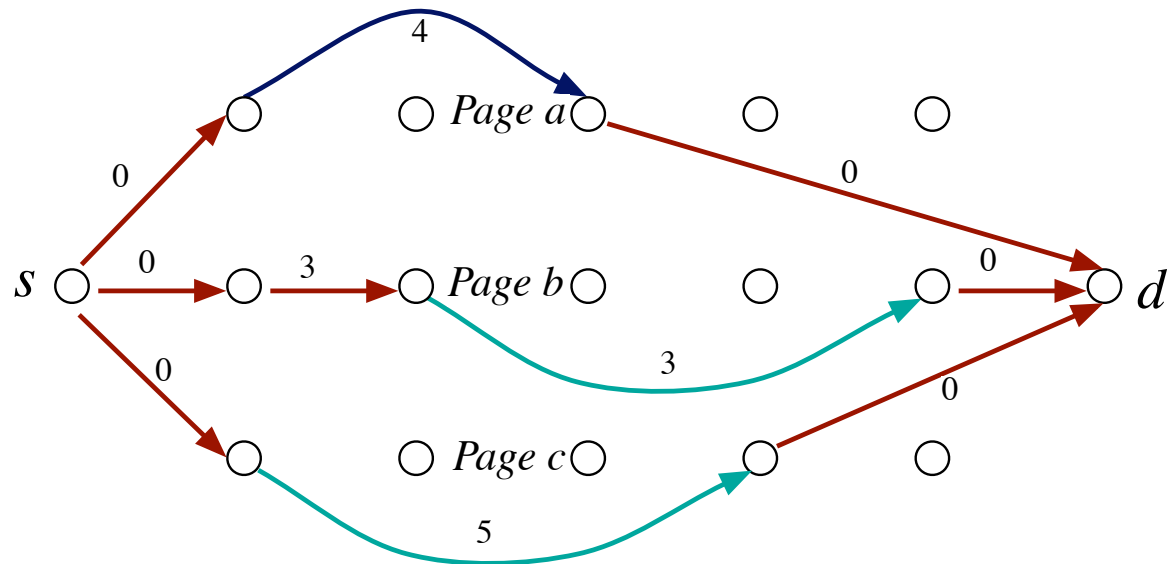
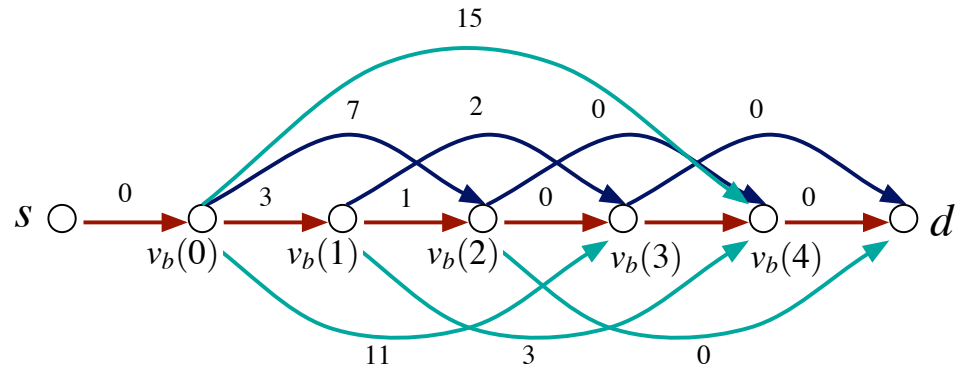
Summary of results

- **ESA00**: 3-speed 3-approx. alg.
- **IPCO02**: 2-speed 2-approx and 4-speed 1-approx
- **FOCS02**: 2-speed 2-approx and 3-speed 1-approx

ESA00: Strategy overview

- Reduction to integer min-cost flow reduction
- Probabilistic interpretation of the LP solution
- Impose a structure on the LP solution to facilitate the definition of a $1/\alpha$ -speed alg. ($\alpha < 1/2$ and $1/\alpha$ is an integer)
- Prove that the algorithm guarantees a factor- $\frac{1}{1-2\alpha}$ approximate solution

Reduction to Min-cost flow



Min-cost flow ILP

$$\min \sum_{t=0}^{T+n} \sum_{t'=t+1}^{T+n} \sum_{i=1}^n w_i(t, t') f_i(t, t') \quad (1)$$

$$\sum_{i=1}^n f_i(-1, 0) = n \quad (2)$$

$$\sum_{t' < t} f_i(t', t) = \sum_{t' > t} f_i(t, t') \quad 1 \leq i \leq n \quad 0 \leq t \leq T+n \quad (3)$$

$$\sum_{t' < t} \sum_{i=1}^n f_i(t', t) \leq 1 \quad 1 \leq t \leq T+n \quad (4)$$

$$f_i(t, t') \in \{0, 1\} \quad 1 \leq i \leq n \quad 0 \leq t < t' \leq T+n \quad (5)$$

$$f_i(t, n+t+1) \in \{0, 1\} \quad 1 \leq i \leq n \quad T+1 \leq t \leq T+n \quad (6)$$

Lemma 1. *The optimal value of IP is the minimum total response time for the broadcast problem.*

Fractional LP solution

- LOPT is n -feasible: fraction of each page can be transmitted in a slot without violating the fourth constraint.

Lemma 2. *All requests are eventually satisfied in the LP solution, equiv.,*

$$R_i(t) > 0 \implies \sum_{0 \leq j \leq t < k \leq T+n} f_i(j, k) = 1$$

- Probabilistic interpretation of LP solution

$$\sum_{t' < t} f_i(t', t) \quad \equiv \quad p_i(t) \quad \Pr\{P_i \text{ is broadcast at } t\}$$

$$\min\{t' > t : p_i(t') \geq 1\} \quad \equiv \quad n_i(t) \quad \text{Requests } (i, t) \text{ are fulfilled by } n_i(t)$$

$$1 - \sum_{j=t}^{n_i(t)-1} p_i(t) \quad \equiv \quad \ell_i(t) \quad \Pr\{\text{Requests } (i, t) \text{ are satisfied at } n_i(t)\}$$

Probabilistic interpretation

- The $R_i(t) > 0$ requests are satisfied at $t < s < n$ with prob. $p_i(s)$
- They are satisfied at $n_i(t)$ with prob. $\ell_i(t)$
- The prob. solution

$$VL = \sum_{i=1}^n \sum_{t=1}^T R_i(t) \left[\sum_{t'=t+1}^{n_i(t)-1} [p_i(t')(t' - t)] + \ell_i(t)(n_i(t) - t) \right]$$

Lemma 3. $VL = LOPT$

Restructuring the fractional solution

- Let $\alpha < 1/2$, $1/\alpha$ is an integer
- They seek a $1/\alpha$ -feasible solution
- $b_i(j) = \min\{t : \sum_{k=1}^t p_i(k) \geq j\alpha\}$, the j^{th} breakpoint for P_i
- $I = \{I_i(j) = [b_i(j), b_i(j+1)] \mid 1 \leq i \leq n, 1 \leq j \leq h(i)\}$
- Interval $I_i(j)$ is satisfied (at time t) if P_i is broadcast at $t \in I_i(j)$

The LEDF algorithm

- Assume all requests up to time t have been scheduled
- To schedule the requests at t : let U be the set of currently unsatisfied intervals and
 1. Pick from U up to $1/\alpha$ intervals with smallest right endpoints
 2. broadcast them at time t
- **Lemma 4.** *Every interval is satisfied in LEDF*

LEDF is a $\frac{1}{1-2\alpha}$ approximation alg.

- **Lemma 5.** *The total response time of the $1/\alpha$ -feasible schedule LEDF is at most $\frac{LOPT}{1-2\alpha}$*
- **Theorem 1.** *There is a polynomial-time 3-speed 3-approx. alg. for the broadcast problem where the objective is to min. the average response time.*

IPCO02: Strategy overview

- ILP formulation
- Relaxed LP solution
- Using LOPT, define a simplified problem instance that allows a $1/\alpha$ -speed alg. ($\alpha \leq 1/2$ and $1/\alpha$ is an integer) .
- The simplified instance can be reduced to min-cost flow on bipartite graph.
- Exact solution to relaxed simplified instance by the integrality theorem.
- Prove the solution for simplified instance is at most $\frac{1}{1-\alpha}$ the optimal solution.

ILP formulation

- $y_{t'}^i = 1$ iff. P_i is broadcast at t'
- $x_{tt'}^i = 1$ iff. requests (i, t) are satisfied at $t' > t$

$$\min \sum_{t=0}^T \sum_{t'=t+1}^{T+n} \sum_{i=1}^n (t' - t) R_i(t) x_{tt'} \quad (7)$$

$$y_{t'}^i - x_{tt'}^i \geq 0 \quad 1 \leq i \leq n, \quad t' > t \quad (8)$$

$$\sum_{t'=t+1}^{T+n} x_{tt'} \geq 1 \quad 1 \leq i \leq n, \quad \forall t \quad (9)$$

$$\sum_i y_{t'}^i \leq 1 \quad \forall t' \quad (10)$$

$$x_{tt'}^i \in \{0, 1\} \quad 1 \leq i \leq n, \quad \forall t, t' \quad (11)$$

$$y_{t'}^i \in \{0, 1\} \quad 1 \leq i \leq n, \quad \forall t' \quad (12)$$

The simplified problem instance

- Let (\mathbf{x}, \mathbf{y}) be the optimal fractional solution
- For $R_i(t) > 0$, define $ft(\alpha, i, t) = \min\{t'' \mid \sum_{t'=t+1}^{t''} x_{t'}^i \geq \alpha\}$
- Transform I to I' by consolidating the requests so that the following property holds for I' :

Property 1. *Let N'_i be the set times of requests for P_i in I' , and $t'_u < t'_v$ then, $\{t, t'\} \subseteq N'_i \implies ft(\alpha, i, t'_u) \leq t'_v$*

- Requests (i, t) grouped with $(i, g(i, t)), g(i, t) \geq t \implies ft(\alpha, i, t) > g(i, t)$

$1/\alpha$ fractional solution for I'

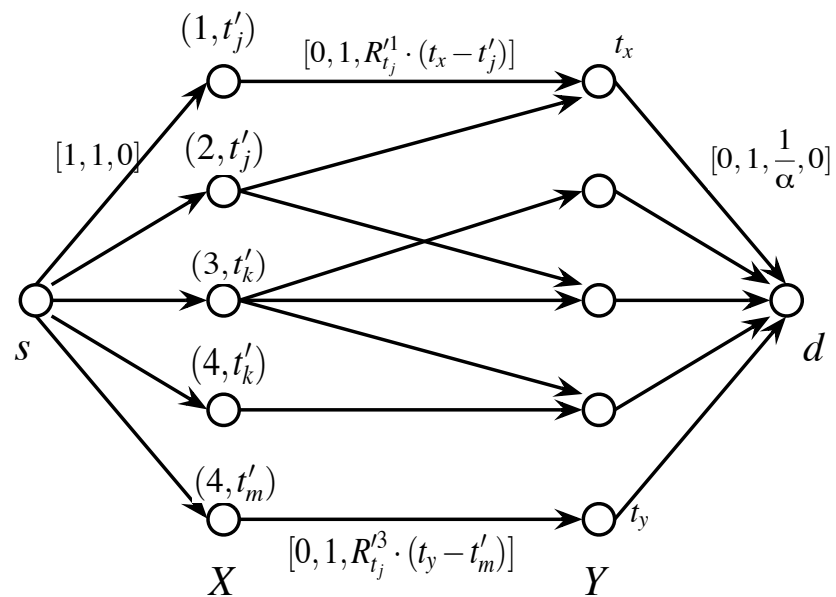
- Recall (\mathbf{x}, \mathbf{y}) is the optimal n -feasible fractional solution for I
- Consider the solution $(\mathbf{x}_\alpha, \mathbf{y})$

$$x_{\alpha t t'}^i = \begin{cases} x_{t t'}^i & \text{if } t' < ft(\alpha, i, t) \\ \alpha - \sum_{t''=t+1}^{t'-1} x_{t t''}^i & \text{if } t' = ft(\alpha, i, t) \\ 0 & \text{otherwise} \end{cases}$$

- The solution $(\frac{1}{\alpha}\mathbf{x}_\alpha, \frac{1}{\alpha}\mathbf{y})$ is $1/\alpha$ -feasible in the weak sense that constraint becomes $\sum_i y_{t'}^i \leq \alpha, \forall t'$.
However, without integer solution $(\mathbf{x}_\alpha, \mathbf{y})$, up to n pages may need to be broadcast each time slot.

$1/\alpha$ integer solution for I'

- Construct the following min. cost flow network N



Analysis

- **Lemma 6.** *The cost of $1/\alpha$ -speed frac. sol. in I' and that of an optimal frac. sol. for I are related as follows*

$$\frac{1}{\alpha} \sum_{(i,t) \in I} R_i(t) \left[\sum_{t'=g(i,t)+1}^{T+n} (t' - g(i,t)) x_{\alpha g(i,t)t'} \right] \leq \frac{1}{1-\alpha} \sum_{(i,t) \in I} R_i(t) (g(i,t) - t)$$

- **Lemma 7.** *For any feasible flow in N there is a $1/\alpha$ -speed feasible fractional sol. for I' of the same cost, and vice-versa*
- **Lemma 8.** *There exists a $1/\alpha$ -speed integral solution for I' of the same cost as the $1/\alpha$ -speed fractional sol. $(\frac{1}{\alpha} \mathbf{x}_\alpha, \frac{1}{\alpha} \mathbf{y})$*
- **Theorem 2.** *There is a $1/\alpha$ -speed, $\frac{1}{1-\alpha}$ approx. solution for the Broadcast Problem.*

FOCS02: Strategy overview

- Same ILP formulation as IPCO02
- Same relaxed LP solution
- Using LOPT, define a simplified problem instance that allows a $1/2$ -speed alg. Difference: don't combine requests, just drop those in the middle .
- The simplified instance can be reduced to min-cost flow on bipartite graph. Different from IPCO02.
- Apply Dependent rounding on the bipartite graph to get a $1/2$ -speed 2-approx. integral sol.
- A 3-speed, 1-approx. algorithm

Dependent Rounding (DR)

- Bipartite graph (A, B, E) with bipartition (A, B)
- $x_{ij} \in [0, 1]$ for each $(i, j) \in E$.
- DR is a poly-time randomized scheme that rounds each $x_{i,j}$ to $X_{i,j} \in \{0, 1\}$ such that the following properties hold

Property 2. Marginal distribution. $\Pr[X_{ij} = 1] = x_{i,j}$

Property 3. Degree preservation. Define fractional degree $d_i = \sum_{j:(i,j) \in E} x_{i,j}$, and integral degree $D_i = \sum_{j:(i,j) \in E} X_{i,j}$, then $\Pr[D_i \in \{\lfloor d_i \rfloor, \lceil d_i \rceil\}] = 1$

Property 4. Negative correlation. Let f be an edge in E , $\forall b \in \{0, 1\}$, $\Pr[\bigwedge_{f \in S} (X_f = b)] \leq \prod_{f \in S} \Pr[X_f = b]$