2006

CS 441 Fall ~~2005~~ Exam 1 ~~for the afternoon section (2:30pm – 3:15pm)~~
There are 6 parts (A to F) on 9 pages with a total score of 110 points. Do all problems.
Calculators are not allowed.

Part A: Propositional and Predicate logics

1. (12 points) Translate each English sentence (a-c) into logic and each logic proposition (d-f) into colloquial English. Let

   P(x): x has a cell phone. C(x,y): x has called y.

   (a) Jill has never called Joe.

   $\neg C(\text{Jill}, \text{Joe})$

   (b) Everybody who has a cell phone has called somebody.

   $\forall x \exists y (P(x) \rightarrow C(x,y))$

   or $\forall x (P(x) \rightarrow \exists y C(x,y))$

   (c) No one has called anybody unless he/she has a cell phone.

   For any x [x has not called anybody unless x has a cellphone].

   For any x [if x does not have a cell phone, then x has not called anybody]

   $\forall x [\neg P(x) \rightarrow$ x has not called anybody]

   $\forall x [\neg P(x) \rightarrow \forall y \neg C(x,y)]$

   or $\forall x \forall y [\neg P(x) \rightarrow \neg C(x,y)]$

   (e) $\exists x \neg P(x)$

   Somebody doesn't have a cell phone.

   (g) $\exists x[\forall y (C(y,x) \rightarrow \neg C(x,y))]$

   There is somebody who never returns a phone call.

   There is somebody who never calls those who has called him.

   (h) $\forall x[P(x) \rightarrow C(\text{Telemarketer},x)]$

   Everybody who has a cell phone has gotten a call from Telemarketer

Part A: (continue)

2. (4 points) Consider the following sentence.

   No one knows the email address of everybody in this class except for Mr. P., who knows all email addresses.

   Define one or more appropriate predicates.
   Translate the sentence into a quantified proposition using the predicate(s).
   Indicate the domain of discourse of each quantifier.

   Let $K(x, y)$ be the statement "x knows y's email address".

   $\neg \exists x \, (x \neq \text{Mr. P.} \wedge \forall y \, K(x, y)) \wedge \forall y \, K(\text{Mr. P.}, y)$

   The domain of all quantifiers is the set of people in this class.

3. (a) (5 points) Construct the truth table for $(p \leftrightarrow \neg q) \wedge (r \wedge p)$.

p	q	r	$\neg q$	$p \leftrightarrow \neg q$	$r \wedge p$	$(p \leftrightarrow \neg q) \wedge (r \wedge p)$
T	T	T	F	F	T	F
T	T	F	F	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	T	F	F	T	F	F
F	F	T	T	F	F	F
F	F	F	T	F	F	F

   (b) (1 point) Classify the above proposition. (circle one)

   tautology **contradiction** **contingency**

Part B: Methods of Proof

1. (8 points) The proof in the table below proves that the following argument is valid. Complete the proof by filling in the inference rule and citation for each line. See the list of inference rules on the last page.

$R(a) \land \neg P(a)$	premise 1
$\forall x\ [\ R(x) \rightarrow (P(x) \lor Q(x))]$	premise 2
$\exists x\ Q(x)$	conclusion

Step	Propositions	Inference rule	Citation
1	$\forall x\ [\ R(x) \rightarrow (P(x) \lor Q(x))]$	premise 2	
2	$R(a) \rightarrow (P(a) \lor Q(a))$	universal instantiation	1
3	$R(a) \land \neg P(a)$	premise 1	
4	$R(a)$	simplification	3
5	$P(a) \lor Q(a)$	modus ponens	2, 4
6	$\neg P(a)$	simplification	3
7	$Q(a)$	disjunctive syllogism	5, 6
8	$\exists x\ Q(x)$	existential generalization	7

2. (10 points) Prove that if $n(n-2)$ is an odd number, then n is an odd number.
Hint: Use one of the following proof strategies: Direct proof, Proof of contrapositive, or Proof by contradiction

Part C: Sets

1. (8 points) Suppose the universal set is U = the set of integers between 1 and 20 inclusive.
   Let A = { 5, 12, 17, 20 }.
   Let B = { x | x is divisible by 4 and $1 \leq x \leq 20$ }.
   Let C = { 2, 3, 7, 11 }.

   Determine the followings.

   (a) B

   $\{ 4, 8, 12, 16, 20 \}$

   (b) $A \cap C$

   $\emptyset$

   (c) $\overline{C} - \overline{A}$

   $\{ 5, 12, 17, 20 \}$

   (d) $|\overline{A}|$

   16

2. (8 points) True/False

   (a) $b \in \{ a, \{ a, b \}, \{ \{ b \} \} \}$ **True** **False** (circle one)

   (b) $\emptyset \cup \{a, b\} \in \{ a, b, c, \emptyset \}$ **True** **False** (circle one)

   (c) $\{ \emptyset \} \subseteq \{ \emptyset, \{ a, b \}, \{ a, \emptyset \} \}$ **True** **False** (circle one)

   (d) If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then it is always the case that $A \cap C = \emptyset$.

   True **False** (circle one)

Part D: Functions

1. (6 points)
   Let A = { u, v, w, x, y }
   Let B = { 1, 2, 3, 4 }
   Let f:A→B where f(u) = 1, f(v) = 4, f(w) = 2, f(x) = 3, and f(y) = 4.

   (a) Let S be the set { v,x,y }. Determine f(S).

   $\{4, 3\}$

   (b) f is a one-to-one function. **True** **False** (circle one)

   (c) f is an onto function. **True** **False** (circle one)

2. (3 points)
   Let g:R→R where g(x) = 2x + 1
   Let h:R→R where h(x) = 3x + 5
   Determine (g ∘ h)(x).

   $(g \circ h)(x) = g(h(x)) = g(3x+5) = 2(3x+5)+1 = 6x+11$

3. (2 points) Let f:R→R such that f(x) = $\lceil x \rceil$ - x. What is the range of f?

   $\{x \mid 0 \leq x < 1\}$

4. (2 points) Suppose f:A→B is a 1-1 function and g:B→C is a bijection. Then g ∘ f is always a bijection.

   True **False** (circle one)

Part E: Sequences and summation

1. (2 points) Write a formula for $\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\ldots+\frac{2}{18}$ using the summation symbol. Do not compute it.

$$\sum_{k=3}^{18} \frac{2}{k} \quad \left(\text{or} \ \sum_{k=1}^{16} \frac{2}{k+2} \right)$$

2. (5 points) Compute the value of $\sum_{i=1}^{3}\sum_{j=1}^{i}(2i+j)$.

$$= \sum_{j=1}^{1} (2\cdot 1 + j) + \sum_{j=1}^{2} (2\cdot 2 + j) + \sum_{j=1}^{3} (2\cdot 3 + j)$$

$$= (2+1) + \left[(4+1) + (4+2)\right] + \left[(6+1) + (6+2) + (6+3)\right]$$

$$= 3 + \left[5+6\right] + \left[7+8+9\right]$$

$$= 3 + 11 + 24$$

$$= 38$$

Part E: (continue)

3. (8 points) Suppose $\{ a_n \}$ is an **arithmetic sequence** with $n \geq 1$ and the first five terms are

   2, 9, 16, 23, 30.

   (a) Compute the 200th term.
   (b) Compute the sum of the first 200 terms.

$$2 \quad 9 \quad 16 \quad 23 \quad 30$$
$$7 \quad 7 \quad 7 \quad 7$$

$a_n = a + nd$ $\qquad$ $d = 7$

$\therefore a_n = a + 7n.$

$2 = a_1 = a + 7 \cdot 1$

$\therefore \quad a = -5$

Thus, $a_n = 7n - 5$ for $n \geq 1$

$a_{200} = 7 \cdot 200 - 5 = 1395$

$$\sum_{k=1}^{200} a_k = \sum_{k=1}^{200} (7k - 5)$$

$$= 7 \sum_{k=1}^{200} k - \sum_{k=1}^{200} 5$$

$$= 7 \times \frac{200(200+1)}{2} - 1{,}000$$

$$= 7 \times 20{,}100 - 1{,}000$$

$$= 140{,}700 - 1{,}000 = 139{,}300$$

Part F: Mathematical Induction and Recursive Definition

1. In this problem you will use mathematical induction to prove the following statement.

   "For any positive integer n, it is the case that $2^n/2 \geq n$."

   Suppose we want to state this sentence as $\forall n$ P(n). Define an appropriate predicate P.

   (2 points) P(n) is the statement " $2^n/2 \geqslant n$ ".

   Basis step:

   (2 points) What is the statement P(1)? $\frac{2^1}{1} \geqslant 1$

   (2 points) Prove the basis step.

   left side $\frac{2^1}{1} = 1$ right side 1

   since left side $\geqslant$ right side. Then P(1) is true

   Inductive Step:

   (2 points) What is the inductive hypothesis?

   $2^k/2 \geqslant k$ (for $k \geqslant 1$)

   (2 points) What do you need to prove in the inductive step?

   $2^{k+1}/2 \geqslant k+1$ (for $k \geqslant 1$)

   (3 points) Complete the inductive step.

$$
\begin{aligned}
\frac{2^{k+1}}{2} &= \frac{2 \times 2^k}{2} \\
&= \frac{2^k + 2^k}{2} \\
&= \frac{2^k}{2} + \frac{2^k}{2} \\
&\geqslant k + k \quad \text{from induction hypothesis} \\
&\geqslant k + 1 \quad \text{because } k \geqslant 1
\end{aligned}
$$

Part F: (continue)

2. (4 points) Let f be a function defined below. Compute f(6).

   $f(0) = 4$
   $f(1) = 5$
   $f(n) = f(n-1) + f(n-2) - 2$ for $n \geq 2$

k	0	1	2	3	4	5	6
$f(k)$	4	5	7	10	15	23	36

$\therefore f(6) = 36$

3. (4 points) Let S be a set defined below. List all elements of S that are smaller than 15.

   $1 \in S$
   If $x \in S$, then $2x \in S$ and $3x \in S$.
   Nothing else is in S.

In S 1, 2, 3, 4, 6, 8, 9, 12

Not in S 5, 7, 10, 11, 13, 14, 15

4. (4 points) Let $\{ a_n \}$ be a sequence where $a_n = 5n+2$ and $n \geq 1$. Give a recursive definition for a_n.

$a_1 = 7$

$a_n = a_{n-1} + 5$ for $n \geqslant 2$