CS 441 Fall 2005 Exam 1 for the evening section (6:00pm - 7:15pm).
There are 8 parts (A to H) on 9 pages with a total score of 117 points. Do all problems.
Calculators are not allowed.

Part A: Propositional and Predicate logics

1. (15 points) Translate each English sentence (a-e) into logic and each logic proposition (f-j) into colloquial English. Let

   H(x) : x is happy. W(x,y) : x works for y.

   (a) Joe doesn't work for Jack.

   $\neg W(Joe, Jack)$

   (b) Someone works for Tom.

   $\exists x \; W(x, Tom)$

   (c) Everyone works for somebody.

   $\forall x \; \exists y \; W(x, y)$

   (d) If Jill is happy, then she works for herself.

   $H(Jill) \rightarrow W(Jill, Jill)$

   (e) Everyone who works for herself is happy.

   $\forall x \, [W(x,x) \rightarrow H(x)]$

   (f) H(Ben) ∧ H(Cindy)

   Both Ben and Cindy are happy.

   (g) ∀x H(x)

   Everybody is happy.

   (h) ¬∃x H(x)

   Nobody is happy.

   (i) ∃x[¬H(x) ∧ W(x, Greg)]

   Some unhappy person works for Greg.

   (j) ∀x[W(x, Bill) → H(x)]

   Everyone who works for Bill is happy.

Part A: (continue)

2. (a) (5 points) Construct the truth table for $(r \wedge \neg p) \vee (q \rightarrow p)$.

p	q	r	$\neg p$	$r \wedge \neg p$	$q \rightarrow p$	$(r \wedge \neg p) \vee (q \rightarrow p)$
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	F	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	T

(b) (1 point) Classify the above proposition. (circle one)

tautology **contradiction** **contingency**

Part B: Methods of Proof

1. (6 points) Prove that the following argument is valid.

$p \vee r$	premise 1
$\neg q$	premise 2
$p \rightarrow q$	premise 3
r	conclusion

Step	Propositions	Inference rule	Citation
1	$p \vee r$	premise 1	
2	$\neg q$	premise 2	
3	$p \rightarrow q$	premise 3	
4	$\neg p$	modus tollens	2, 3
5	r	disjunctive syllogism	1, 4

2. (10 points) Prove that if (n-2)(n+2) is an even number, then n is an even number.

Use indirect proof.

Thus, we will prove that if n is odd, then $(n-2)(n+2)$ is odd.

Assume that n is an odd number.

Thus, $n = 2k-1$ for some integer k.

Now consider $(n-2)(n+2)$

$$
\begin{aligned}
(n-2)(n+2) &= (2k-3)(2k+1) \\
&= 4k^2 - 6k + 2k - 3 \\
&= 4k^2 - 4k - 3 \\
&= 2(2k^2 - 2k - 1) - 1
\end{aligned}
$$

Thus, $(n-2)(n+2) = 2q$ where q is an integer.

In particular, $q = 2k^2 - 2k - 1$.

Thus, $(n-2)(n+2)$ is an odd number.

Thus, if n is odd, then $(n-2)(n+2)$ is odd.

Equivalently, if $(n-2)(n+2)$ is even, then n is even.

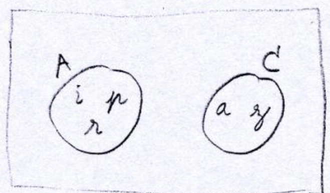
Part C: Sets

1. (8 points) Suppose the universal set is U = { x | x is a lower case letter in the English alphabet }.
   Let A = { i, p, r }.
   Let B = { x | x is a vowel $\wedge$ x follows 'g' in the alphabet }.
   Let C = { a, z }.

   Represent each of the sets by listing their members.

   (a) B = { i, o, u }

   (b) $A \cap C = \emptyset$

   (c) $\bar{C} - \bar{A}$ = { i, p, r }

   (d) $|\bar{C}|$ = 24

2. (8 points) True/False

   (c) $c \subseteq \{ a, b, \{ c, \emptyset \} \}$ **True** **False** (circle one)

   (a) $\{ b, a \} \subseteq \{ b, \{b, a\}, \{a\} \}$ **True** **False** (circle one)

   (b) $\emptyset \in \{ c, \{ \emptyset \}, \{ c \} \}$ **True** **False** (circle one)

   (d) Suppose A = { 3, 3, 4, 3, { 5, 2, $\emptyset$ } }. Then | A | = 3

   = { 3, 4, { 5, 2, $\emptyset$ } }

Part D: Functions

1. (6 points)
   Let A = { 1, 2, 3, 4 }
   Let B = { u, v, w, x, y }
   Let f:A→B where f(1) = x, f(2) = y, f(3) = u, f(4) = v.

   (a) Determine f({ 2,3,4 }).

   $\{y, u, v\}$

   (b) f is a one-to-one function. **True** **False** (circle one)

   (c) f is an onto function. **True** **False** (circle one)

2. (3 points)
   Let g:R→R where g(x) = 3x + 2
   Let h:R→R where h(x) = 2x - 3
   Determine (g ∘ h)(x).

$$\begin{aligned}(g\circ h)(x) &= g(h(x))\\ &= g(2x-3)\\ &= 3(2x-3)+2\\ &= 6x-9+2\\ &= 6x-7\end{aligned}$$

3. (2 points) Let f:R→R such that f(x) = ⌈x⌉ - ⌊x⌋. What is the range of f?

$f(x) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ 1 & \text{if } x \text{ is not an integer} \end{cases}$ Thus, the range of f is $\{0, 1\}$.

4. (2 points) Suppose f:A→B is a bijection and g:B→C is an onto function. Then g ∘ f is an onto function.

   True **False** (circle one)

Part E: Big-O notation

1. Give the best big-O estimate for each of these functions.

   (a) (2 points) $7n^3 + 3n + 4$ $\qquad O(n^3)$

   (b) (2 points) $(2n + 3\log n)(n^2 + 2n)$ $\qquad O(n^3)$

   (c) (3 points) $\underbrace{3n^2 \log n}_{O(n^2 \log n)} + \underbrace{0.02n^3}_{O(n^3)} + \underbrace{(2n^2 + 4)}_{O(n^2)}\underbrace{(n^2 + 12)}_{O(n^2)}$ $\qquad O(n^4)$

2. (2 points) If $f(n) = \Omega(g(n))$ and $h(n) = O(g(n))$, then $h(n) = O(f(n))$.

   True **False** (circle one)

3. (4 points) Show that $3n^3+4n^2+5$ is $O(n^3)$. You should show systematically how to obtain a pair of witnesses (C and k) for this relationship.

Note that $4n^2 \leq 4n^2 \cdot n$ if $n \geq 1$

and $5 \leq 5 \cdot n^3$ if $n \geq 1$

Thus, $3n^3 + 4n^2 + 5 \leq 3n^3 + 4n^3 + 5$ if $n \geq 1$

$\leq 3n^3 + 4n^3 + 5n^3$ if $n \geq 1$

$= \underbrace{12}_{C} n^3$ if $n \geq \underbrace{1}_{k}$

Thus, $C = 12$ and $k = 1$ is a pair of witnesses to the relationship $3n^3 + 4n^2 + 5$ being $O(n^3)$.

Part F: Sequences and summation

1. (2 points) Write a formula using the summation symbol for "the sum of the first n even positive integers".

$$\sum_{i=1}^{n} 2i$$

2. (8 points) Suppose $\{ a_n \}$ is an **arithmetic sequence** with $n \geq 1$ and the first five terms are

   5, 11, 17, 23, 29. $\rightarrow a_n = bn + c$ where b, c are constants

   (a) Compute the 200th term.
   (b) Compute the sum of the first 200 terms.

n	1	2	3	4	5
a_n	5	11	17	23	29

6 6 6 6

The difference between adjacent terms is 6.

$$\therefore\ a_n = 6n + C$$

$$5 = a_1 = 6 \cdot 1 + C \quad \Rightarrow \quad C = -1$$

$$\therefore\ a_n = 6n - 1$$

(a) $a_{200} = 6 \cdot 200 - 1 = 1199$

(b) $$\sum_{i=1}^{200} a_i = \sum_{i=1}^{200} (6i - 1) = \left(\sum_{i=1}^{200} 6i\right) - \left(\sum_{i=1}^{200} 1\right)$$

$$= 6\left(\sum_{i=1}^{200} i\right) - 200$$

$$= 6 \cdot \frac{200 \cdot 201}{2} - 200 \quad \text{because } \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$= 120400$$

Part G: Recursive Definition

1. (4 points) Let f be a function defined below. Compute f(6).

   $f(1) = 2$
   $f(2) = 4$
   $f(n) = 2\,f(n\text{-}1) - f(n\text{-}2)$ for $n \geq 3$

n	1	2	3	4	5	6
$f(n)$	2	4	6	8	10	12

$f(6) = 12$

2. (4 points) Let S be a set defined below. Describe the set S in simple English.

   $2 \in S$
   $3 \in S$
   If $x \in S$ and $y \in S$, then $x+y \in S$.
   Nothing else is in S.

S is the set of all integers greater than 1.

$S = \{x \in Z \mid x \geqslant 2\}$

$S = Z^+ - \{1\}$

3. (4 points) Let $\{ a_n \}$ be a sequence where $a_n = 4n+5$ and $n \geq 1$. Give a recursive definition for a_n.

$a_1 = 9$

$a_n = a_{n-1} + 4$ for $n \geqslant 2$

Part H: Mathematical Induction

1. (3 points)To prove $\forall n$ P(n) using mathematical induction, what two statements do we need to prove?

$P(1)$ and $\forall k\, [P(k) \rightarrow P(k+1)]$

2. Prove that the sum of the first n even positive integers is $n(n+1)$ using mathematical induction.

Give a predicate P(n) for the statement to be proved.

(2 points) P(n) means $\sum_{i=1}^{n} 2i = n(n+1)$

Basis step:

(2 points) What is the statement that we need to prove in the basis step?

$P(1)$ $\sum_{i=1}^{1} 2i = 1(1+1)$

(2 points) Prove this statement.

$\sum_{i=1}^{1} 2i = 2 \cdot 1 = 2$ $1(1+1) = 2$

equal

Thus $\sum_{i=1}^{1} 2i = 1(1+1)$

Inductive step:

(2 points) What is the inductive hypothesis? (what do we assume in the inductive step?)

$P(k)$ $\sum_{i=1}^{k} 2i = k(k+1)$

(2 points) What is the statement that we need to prove using the inductive hypothesis?

$P(k+1)$ $\sum_{i=1}^{k+1} 2i = (k+1)(k+2)$

(3 points) Prove this statement.

$$\sum_{i=1}^{k+1} 2i = \left(\sum_{i=1}^{k} 2i\right) + 2(k+1)$$

$$= k(k+1) + 2(k+1)$$ from the inductive hypothesis

$$= (k+1)(k+2)$$