CS 684: Algorithmic Game Theory Instructor: Éva Tardos	Scribe: Jonathan Winter Monday February 9, 2004

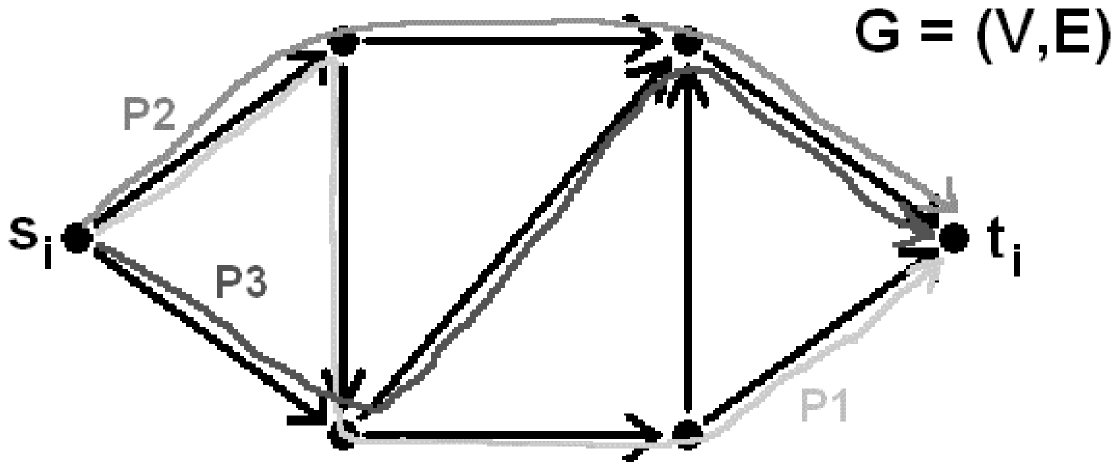
Worst Case Nash / Optimal Ratio

What we learned from convex optimization:

APPLICATION:

- Let G = (V, E), with a continuous and monotonic delay function, $d_e(x) \geq 0$, for each edge $e \in E$.
- Let s_i, t_i be source, sink pairs for i = 1, 2, …, k and $\boldsymbol{P}_i$ = {set of all paths from s_i to t_i}.
- Define flow $f_P \geq 0$ for $P \in \bigcup_i \boldsymbol{P}_i$ with the property that $\sum_{P \in P_i} f_P = 1$.

Now, the flow on edge e is $f(e) = \sum_{P,\, e \in P} f_P$. Also, delay on P is $d_P(f) = \sum_{e \in P} d_e(f(e))$.



The flow at Nash Equilibrium requires that $\forall\ P \in \boldsymbol{P}_i$, if $f_P > 0$ and $Q \in \boldsymbol{P}_i$, then

$$d_P(f) \leq d_Q(f). \qquad \text{(A)}$$

(The logic behind this is that no user on a path P wants to switch to any other path.)

THEOREM 7.1: Suppose the goal is to minimize $\sum_{e \in E} c_e(f(e))$, where c_e is convex and differentiable. (Note: the summation is separable.) Then, the flow f is optimal if and only if $\forall\ P \in \boldsymbol{P}_i$, if $f_P > 0$ and $Q \in \boldsymbol{P}_i$, then

$$\sum_{e \in P} c_e'(f(e)) \leq \sum_{e \in Q} c_e'(f(e)). \qquad \text{(B)}$$

(Note: here c_e' is the derivative of c_e.)

COROLLARY 7.1a: Nash Equilibrium is the optimal flow and of course optimizes $\text{Ø}(f)$, where $\text{Ø}(f) = \sum_{e \in E} \int_0^{f(e)} d_e(x)\, dx$. This follows by substituting

$_0\int^{f(e)} d_e(x)\,dx$ for $c_e(f(e))$ in equation (B) and noting that the derivative of the integral, $d/dx\,(_0\int^{f(e)} d_e(x)\,dx) = d_e(f(e))$. The resulting equation is

$$\sum_{e\in P} d_e(f(e)) \leq \sum_{e\in Q} d_e(f(e)),$$

which is equivalent to $d_P(f) \leq d_Q(f)$. Hence, by (A) our Nash Equilibrium flow satisfies the preconditions for Theorem 7.1.

COROLLARY 7.1b: The approximate Nash Equilibrium flow can be found in polynomial time.

Let us consider a new objective function, $\sum_{P\in \boldsymbol{P}_i} f_P \cdot d_P(f_P) = \sum_{e\in E} f(e) \cdot d_e(f(e))$. Assume $x \cdot d_e(x)$ is convex for all edges (this is usually true for most $d_e(x)$ functions).

COROLLARY 7.1c: If $x \cdot d_e(x)$ is convex for all $e \in E$, then the optimal flow, f, is obtained if and only if $\forall\, P \in \boldsymbol{P}_i$, if $f_P > 0$ and $Q \in \boldsymbol{P}_i$, then

$$\sum_{e\in P} (d_e(f(e)) + f(e) \cdot d_e'(f(e))) \leq \sum_{e\in Q} (d_e(f(e)) + f(e) \cdot d_e'(f(e))) \qquad \text{(C)}$$

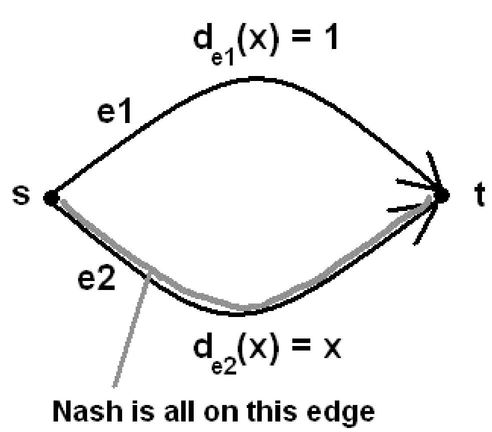
COROLLARY 7.1d: The approximate optimal flow (in an average happiness sense) can be computed if $x \cdot d_e(x)$ is convex.

COROLLARY 7.1e: For a new delay function $\boldsymbol{d}_e^*(x) = d_e(x) + x \cdot d_e'(x)$, the Nash Equilibrium flow is actually the optimal flow (in an average happiness sense) for the original routing problem. Therefore, a network administrator's strategy to achieve optimal flow could be to charge $x \cdot d_e'(x)$ as a tax/fee for using the network.

** Charging money can make people behave Nashfully. **

GOAL: Compare Nash flow with Optimal flow:

Example 1:



Nash: All flow is on lower edge with delay $d_{e2}(1) = 1$.

Optimal:

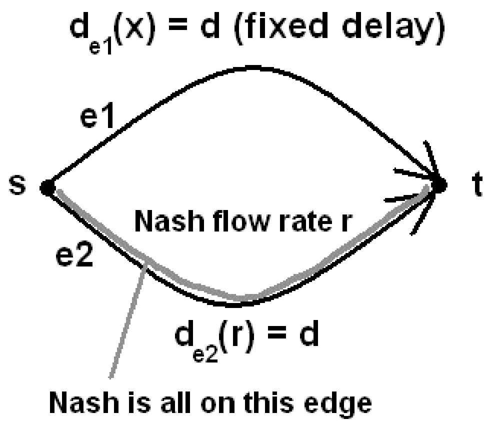
Upper edge: $\boldsymbol{d}_{e1}^*(x) = d_{e1}(x) + x \cdot d_{e1}'(x) = 1 + x \cdot 0 = 1$

Lower edge: $\boldsymbol{d}_{e2}^*(x) = d_{e2}(x) + x \cdot d_{e2}'(x) = x + x \cdot 1 = 2x$

Optimal occurs when delays are equal

($\boldsymbol{d}_{e1}*(x) = \boldsymbol{d}_{e2}*(x)$), so the flow will be split ½ on the top edge and ½ on the bottom edge.

Example 2:



Nash: All flow is on lower edge with delay $d_{e2}(r) = d$, where r is the Nash flow rate.

Optimal:

Upper edge: $\boldsymbol{d}_{e1}*(x) = d_{e1}(x) + x \cdot d_{e1}'(x)$

$$= d_{e2}(r) + x \cdot d/dx\,(d_{e2}(r))$$
$$= d + x \cdot 0$$
$$= d$$

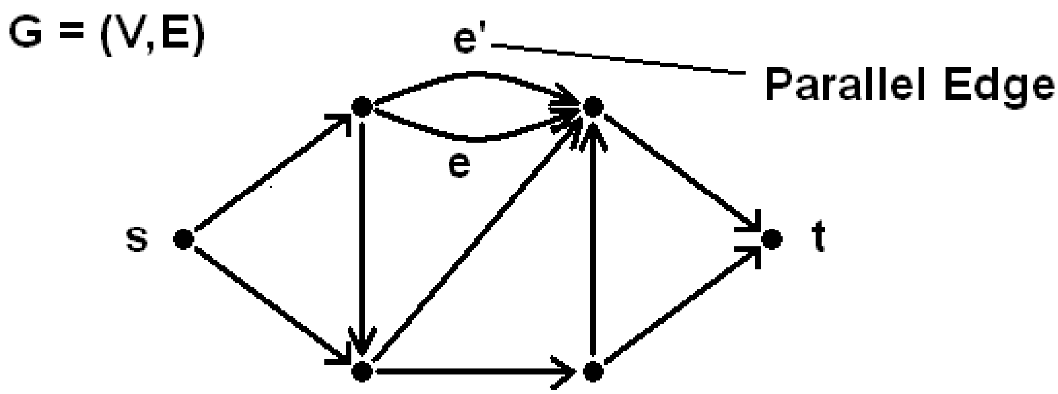
Lower edge: $\boldsymbol{d}_{e2}*(x) = d_{e2}(x) + x \cdot d_{e2}'(x)$

If r* is the flow on e2 in the optimal case, r – r* will be the flow on e1. Then, r* can be computed by solving:

$$d = \boldsymbol{d}_{e2}*(r^*) = d_{e2}(r^*) + r^* \cdot d_{e2}'(r^*).$$

THEOREM 7.2 (Roughgarden): The worst case of Nash / Optimal ratio for any class of delays, $x \cdot d_e(x)$ (convex and differentiable), is on a 2-edge, 2 node graph with one edge having a constant delay.

PROOF: Consider the graph G = (V, E) as shown.



Let f^N be the Nash flow on G. Consider G' = (V, E') created from G by adding a parallel copy to every edge $e \in E$ called e'. Let e' have fixed delay $d_{e'}(x) = d_e(f^N(e))$.

Facts:

1. f^N is still a Nash flow for G'.
2. The Optimal flow for G' may have improved over the Optimal flow for the original graph G.
3. We claim that the Optimal flow on G' is obtained from the Nash by dividing the flow between e and e' optimally as shown in Example 2.

Proof of 3: Assume $\boldsymbol{f}^*$ is the flow constructed in Claim 3 by dividing the flow $f^N(e)$ between the two parallel copies. Let $d_{e'}(x)$ to denote the (constant) delay of e', the new parallel copy of edge e. We want to claim that $\boldsymbol{f}^*$ is the optimal flow. Define the new delay function as $\boldsymbol{d}_e^*(x) = d_e(x) + x \cdot d_e'(x)$. By definition of how we divide the flow between the two copies of an edge, e and e', we have the following:

$$\boldsymbol{d}_e^*(\boldsymbol{f}^*(e)) = \boldsymbol{d}_{e'}^*(\boldsymbol{f}^*(e)) = d_e(f^N(e))$$

Therefore, $\boldsymbol{f}^*$ is the Nash flow subject to the delay function $\boldsymbol{d}_e^*$ (all flow on the shortest s_i - t_i paths). This implies that $\boldsymbol{f}^*$ is the Optimal flow for G'.

Continuing with the proof of Theorem 7.2:

$$\frac{\text{Nash}}{\text{Opt}} \text{ (on G)} \quad \leq \quad \frac{\text{cost of } f^N}{\text{cost of } \boldsymbol{f}^*} = \frac{\sum_{e \in E} f^N(e) \cdot d_e(f^N(e))}{\sum_{e \in E'} \boldsymbol{f}^*(e) \cdot d_e(\boldsymbol{f}^*(e))}$$

$$\leq \max_e \frac{f^N(e) \cdot d_e(f^N(e))}{\boldsymbol{f}^*(e) \cdot d_e(\boldsymbol{f}^*(e)) + \boldsymbol{f}^*(e') \cdot d_{e'}(\boldsymbol{f}^*(e'))}$$

Notes: The first inequality follows from applying facts 1, 2, and 3 to G'.
The final inequality follows from the math theorem: $\frac{a + b}{a' + b'} \leq \max\left(\frac{a}{a'}, \frac{b}{b'}\right)$.