CS 441 Discrete Mathematics for CS
Lecture 6

Predicate logic
Translation II.

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Predicate logic

- Explicitly models objects and their properties
- Allows to make quantified statements

Basic building blocks of the predicate logic:

- **Constant** –models a specific object
  Examples: "John", "France", "7"
- **Variable** – represents object of specific type **(defined by the universe of discourse)**
  Examples: x, y
  (universe of discourse can be people, students, cars)
- **Predicate -** over one, two or many variables or constants.
  Examples: Red(car23), student(x), married(John,Ann)

Predicates

Predicates represent properties or relations among objects

A predicate, say P(x), assigns a value **true or false** to each x depending on whether the property holds or not for a specific x.

Example:

- Assume **Student(x)**
  - Student(John) …. T (if John is a student)
  - Student(Ann) …. T (if Ann is indeed a student)
  - Student(Jane) ….. F (if Jane is not a student)
  - …
- Student(x) is **not a proposition,**
- Student(Ann) **is a proposition**
- ∀x Student(x) is **a proposition**
- ∃x Student(x) is **a proposition**

Translation with quantifiers

Sentence:

- All Upitt students are smart.

- **Assume:** the domain of discourse of x are Upitt students
- **Translation:**
- ∀x Smart(x)

- **Assume:** the universe of discourse are students (all students):
- ∀x at(x,Upitt) → Smart(x)

- **Assume:** the universe of discourse are people:
- ∀x student(x) ∧ at(x,Upitt) → Smart(x)

Translation with quantifiers

Sentence:

- Someone at CMU is smart.

- **Assume:** the domain of discourse are all CMU affiliates
- **Translation:**
- $\exists$ x Smart(x)

- **Assume:** the universe of discourse are people:
- $\exists$ x at(x,CMU) $\wedge$ Smart(x)

Translation with quantifiers

- **Assume two predicates S(x) and P(x)**

Universal statements typically tie with implications

- **All S(x) is P(x)**
  - $\forall$x (S(x) $\rightarrow$ P(x))
- **No S(x) is P(x)**
  - $\forall$x(S(x) $\rightarrow$ $\neg$P(x))

Existential statements typically tie with conjunction

- **Some S(x) is P(x)**
  - $\exists$x (S(x) $\wedge$ P(x))
- **Some S(x) is not P(x)**
  - $\exists$x (S(x) $\wedge$ $\neg$P(x))

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- Every real number has its corresponding negative.
- **Translation:**
  - Assume:
    - a real number is denoted as x and its negative as y
    - A predicate P(x,y) denotes: “x + y =0”
- Then we can write:

  $\forall x \; \exists \, y \; P(x,y)$

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- There is a person who loves everybody.
- **Translation:**
  - Assume:
    - Variables x and y denote people
    - A predicate L(x,y) denotes: “x loves y”
- Then we can write in the predicate logic:

  ?

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- There is a person who loves everybody.
- **Translation:**
  - Assume:
    - Variables x and y denote people
    - A predicate L(x,y) denotes: "x loves y"
- Then we can write in the predicate logic:

  $\exists x \forall y \; L(x,y)$

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. ?

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
- Everybody loves somebody. ?

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\ L(x,y)$
- There is somebody whom everybody loves. ?

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\, L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\, L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\, L(x,y)$
- There is somebody who Raymond doesn't love. ?

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\, L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\, L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\, L(x,y)$
- There is somebody who Raymond doesn't love.
  $\exists y \neg L(Raymond,y)$
- There is somebody whom no one loves. ?

Translation exercise

Suppose:

- Variables x,y denote people
- L(x,y) denotes "x loves y".

Translate:

- Everybody loves Raymond. $\forall x\ L(x,Raymond)$
- Everybody loves somebody. $\forall x \exists y\ L(x,y)$
- There is somebody whom everybody loves. $\exists y \forall x\ L(x,y)$
- There is somebody who Raymond doesn't love.
  $\exists y \neg L(Raymond,y)$
- There is somebody whom no one loves.
  $\exists y\ \forall x\ \neg L(x,y)$

Order of quantifiers

The order of nested quantifiers matters if quantifiers are of different type

- $\forall x \exists y\ L(x,y)$ is not the same as $\exists y \forall x\ L(x,y)$

Example:

- Assume L(x,y) denotes "x loves y"

- Then: $\forall x \exists y\ L(x,y)$
- Translates to: Everybody loves somebody.
- And: $\exists y\ \forall x\ L(x,y)$
- Translates to: ?

Order of quantifiers

The order of nested quantifiers matters if quantifiers are of different type

- $\forall x \exists y\ L(x,y)$ is not the same as $\exists y \forall x\ L(x,y)$

Example:

- Assume L(x,y) denotes "x loves y"
- Then: $\forall x \exists y\ L(x,y)$
- Translates to: Everybody loves somebody.
- And: $\exists y\ \forall x\ L(x,y)$
- Translates to: There is someone who is loved by everyone.

The meaning of the two is different.

Order of quantifiers

The order of nested quantifiers does not matter if quantifiers are of the same type

Example:

- For all x and y, if x is a parent of y then y is a child of x
- **Assume:**
  - Parent(x,y) denotes "x is a parent of y"
  - Child(x,y) denotes "x is a child of y"
- Two equivalent ways to represent the statement:
  - $\forall x\ \forall y\ Parent(x,y) \rightarrow Child(y,x)$
  - $\forall y\ \forall x\ Parent(x,y) \rightarrow Child(y,x)$

Negation of quantifiers

English statement:

- Nothing is perfect.
- **Translation:** $\neg \exists x \, Perfect(x)$

Another way to express the same meaning:

- **Everything ...**

Negation of quantifiers

English statement:

- Nothing is perfect.
- **Translation:** $\neg \exists x \, Perfect(x)$

Another way to express the same meaning:

- **Everything is imperfect.**
- **Translation:** ?

Negation of quantifiers

English statement:

- Nothing is perfect.
- **Translation:** $\neg \exists x$ Perfect(x)

Another way to express the same meaning:

- **Everything is imperfect.**
- **Translation:** $\forall x \neg$ Perfect(x)

Conclusion: $\neg \exists x\, P(x)$ is equivalent to $\forall x \neg P(x)$

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** ?

Another way to express the same meaning:

- There is a dog that …

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** $\neg \forall x \; Dog(x) \rightarrow Fleabag(x)$

Another way to express the same meaning:

- There is a dog that …

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** $\neg \forall x \; Dog(x) \rightarrow Fleabag(x)$

Another way to express the same meaning:

- There is a dog that is not a fleabag.
- **Translation:** ?

Negation of quantifiers

English statement:

- It is not the case that all dogs are fleabags.
- **Translation:** $\neg \forall x \ Dog(x) \rightarrow Fleabag(x)$

Another way to express the same meaning:

- There is a dog that is not a fleabag.
- **Translation:** $\exists x \ Dog(x) \wedge \neg Fleabag(x)$

- Logically equivalent to:
  - $\exists x \neg (Dog(x) \rightarrow Fleabag(x))$

Conclusion: $\neg \forall x \ P(x)$ is equivalent to $\exists x \neg P(x)$

Negation of quantified statements

Negation	Equivalent
$\neg\exists x \ P(x)$	$\forall x \ \neg P(x)$
$\neg\forall x \ P(x)$	$\exists x \ \neg P(x)$