

**CS 441 Discrete Mathematics for CS
Lecture 30**

Probabilities

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Course administration

- **Homework assignment**
 - no assignment is due this week
- **Recitation today**
 - **Midterm solutions**

Course web page:

<http://www.cs.pitt.edu/~milos/courses/cs441/>

Probability

Discrete probability theory.

- Used to compute the odds of seeing some outcomes. Related to counting when the outcomes are equally likely.

Example: Coin flip

- Assume 2 outcomes (head and tail) and each of them is equally likely
- Odds: 50%, 50%
- the probability of seeing:
 - a head is 0.5
 - a tail is 0.5

Probabilities

- **Experiment:** a procedure that yields one of the possible outcomes
- **Sample space:** a set of possible outcomes
- **Event:** a subset of possible outcomes (E is a subset of S)
- **Assuming the outcomes are equally likely, the probability of an event E, defined by a subset of outcomes from the sample space S is**
 - $P(\text{Event}) = |E| / |S|$
- The cardinality of the subset divided by the cardinality of the sample space.

Probabilities

Example 1:

- A box with 4 red balls and 6 blue balls. What is the probability that we pull the red ball out.
- $P(E) = ?$

Probabilities

Example 1:

- A box with 4 red balls and 6 blue balls. What is the probability that we pull the red ball out.
- $P(E) = 4/10 = 0.4$

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes: ?

Probabilities

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- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
- $(1,6) (2,6) \dots (6,1), \dots (6,6)$ total: 36

- Outcomes leading to 7?

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
 - $(1,6) (2,6) \dots (6,1), \dots (6,6)$ total: 36
- Outcomes leading to
 - $(1,6) (2,5) \dots (6,1)$ total: 6
 - $P(\text{sum}=7) = ?$

Probabilities

Example 2:

- roll of two dices.
- What is the probability that the outcome is 7.
- Possible outcomes:
 - $(1,6) (2,6) \dots (6,1), \dots (6,6)$ total: 36
- Outcomes leading to
 - $(1,6) (2,5) \dots (6,1)$ total: 6
 - $P(\text{sum}=7) = 6/36 = 1/6$

Probabilities

More complex:

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes: ?

Probabilities

More complex:

- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes:
 - $C(40,6) = \dots$
- Probability of winning: ?
 - $P(E) = ?$

Probabilities

More complex:

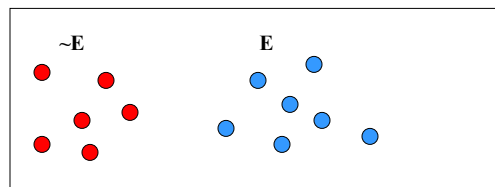
- Odd of winning a lottery: 6 numbers out of 40.
- Total number of outcomes:
 - $C(40,6) = \dots$
- Probability of winning: ?
 - $P(E) = 1/C(40,6) = 34! \cdot 6! / 40! = 3,838,380$

Probabilities

Theorem: Let E be an event and $\sim E$ its complement with regard to S . Then:

- $P(\sim E) = 1 - P(E)$

Sample space

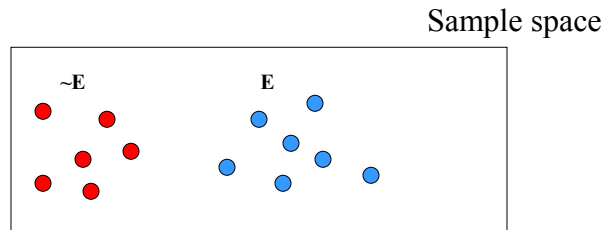


Proof.

Probabilities

Theorem: Let E be an event and $\sim E$ its complement with regard to S . Then:

- $P(\sim E) = 1 - P(E)$



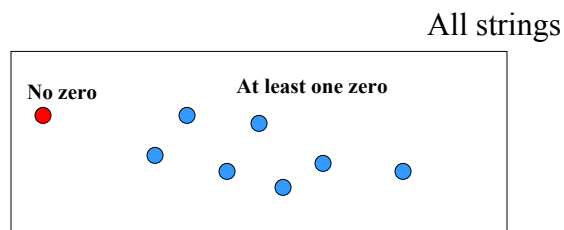
Proof.

$$P(\sim E) = (|S| - |E|) / |S| = 1 - |E| / |S|$$

Probabilities

Example:

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

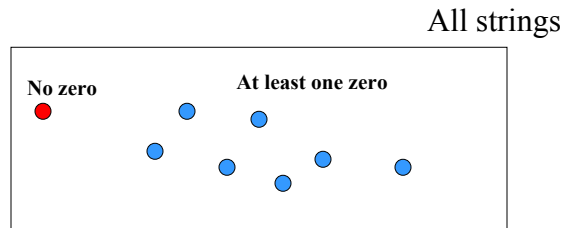


- Event: seeing all zero string
- \sim Event: seeing at least one string

Probabilities

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- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

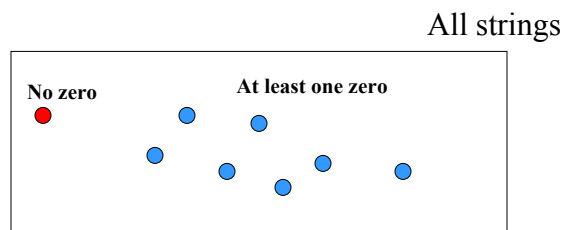


- Event: seeing all zero string $P(E) = ?$
- \sim Event: seeing at least one string

Probabilities

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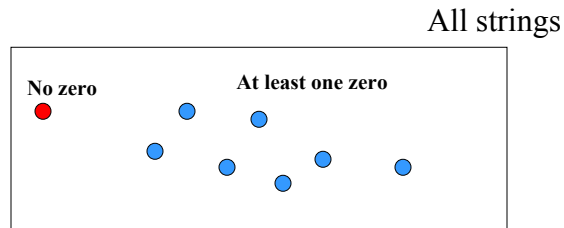


- Event: seeing all zero string $P(E) = 1/2^{10}$
- \sim Event: seeing at least one string $P(\sim E) = ?$

Probabilities

Example:

- 10 randomly generated bits. What is the probability that there is at least one zero in the string.

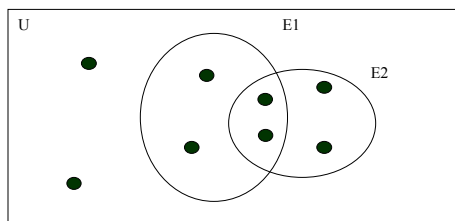


- Event: seeing all zero string $P(E) = 1/2^{10}$
- \sim Event: seeing at least one string $P(\sim E) = 1 - P(E) = 1 - 1/2^{10}$

Probabilities

Theorem. Let E_1 and E_2 be two events in the sample space S .
Then:

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$
- This is an example of the inclusion-exclusion principle



Probabilities

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Then:

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

Example: Probability that a positive integer ≤ 100 is divisible either by 2 or 5.

- E_1 : the integer is divisible by 2, E_2 : the integer is divisible by 5
- $P(E_1) = ?$

Probabilities

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Then:

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Example: Probability that a positive integer ≤ 100 is divisible either by 2 or 5.

- $P(E_1) = 50/100$
- $P(E_2) = ?$

Probabilities

Theorem. Let E_1 and E_2 be two events in the sample space S .

Then:

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \text{ and } E_2)$

Example: Probability that a positive integer ≤ 100 is divisible either by 2 or 5.

- $P(E_1) = 50/100$
- $P(E_2) = 20/100$
- $P(E_1 \text{ and } E_2) = ?$

Probabilities

Theorem. Let E_1 and E_2 be two events in the sample space S .

Then:

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- $P(E_1) = 50/100$
- $P(E_2) = 20/100$
- $P(E_1 \text{ and } E_2) = 10/100$
- $P(E_1 \cup E_2) = (5+2-1)/10 = 6/10$

Probabilities

- Assumption applied so far:
 - **the probabilities of each outcome are equally likely.**
- However in many cases outcome may not be equally likely.

Example: a biased coin or a biased dice.

- Probability of head 0.6, probability of a tail 0.4.

Probabilities

Axioms of the probability theory:

- Probability of a discrete outcome is:
 - $0 \leq p(s) \leq 1$
- Sum of probabilities over all outcomes is = 1
- For any two events E1 and E2 holds:
 $P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \text{ and } E2)$

Probabilities

Definition: A function $p: S \rightarrow [0,1]$ satisfying a condition is called a **probability distribution**.

A **uniform distribution** is a special case: assigns an equal probability to each outcome.

Example: a biased coin.

- Probability of head 0.6, probability of a tail 0.4

Probability distribution:

- Head \rightarrow 0.6 The sum of the two probabilities sums to 1
- Tail \rightarrow 0.4

Conditional probability

Definition: Let E and F be two events such that $P(F) > 0$. The **conditional probability** of E given F

- $P(E|F) = P(E \text{ and } F) / P(F)$

- **Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. ?**

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- **Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- **$P(BB) = ?$**

Conditional probability

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- **Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- $P(\text{BB}) = 1/4$
- $P(\text{one boy}) = ?$

Conditional probability

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- **Example:**
- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- $P(\text{BB}) = 1/4$
- $P(\text{one boy}) = 3/4$
- $P(\text{BB}|\text{given a boy}) = ?$

Conditional probability

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Example:

- What is the probability that a family has two boys given that they have at least one boy. Assume the probability of having a girl or a boy is equal.
- **Possibilities. BB BG GB GG**
- $P(\text{BB}) = 1/4$
- $P(\text{one boy}) = 3/4$
- $P(\text{BB}|\text{given a boy}) = 1/4 / 3/4 = 1/3$

Independence

Definition: The events E and F are said to be **independent** if:

- $P(E \text{ and } F) = P(E)P(F)$

Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos =

Independence

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- All combos = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}
the number of elements = ?

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Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}
the number of elements = 8
- Both sexes = ?

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Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = ?

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- $P(E \text{ and } F) = P(E)P(F)$

Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = ?

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Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- $P(E \text{ and } F) = ?$

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- All combos = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- $P(E \text{ and } F) = 3/8$ and $P(E)*P(F) = ?$

Independence

Definition: The events E and F are said to be **independent** if:

- $P(E \text{ and } F) = P(E)P(F)$

Example. Assume that E denotes the family has three children of both sexes and F the fact that the family has at most one boy. Are E and F independent?

- All combos = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG} the number of elements = 8
- Both sexes = {BBG BGB GBB BGG GBG GGB} # = 6
- At most one boy = {GGG GGB GBG BGG} # = 4
- E and F = {GGB GBG BGG} # = 3
- $P(E \text{ and } F) = 3/8$ and $P(E)*P(F) = 4/8 * 6/8 = 3/8$
- **The two probabilities are equal \rightarrow E and F are independent**