CS 3750 Machine Learning

Lecture 2

Graphical models

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Graphical models

Aim to represent complex multivariate probabilistic models

- **multivariate -> cover multiple random variables**

$$P(\mathbf{X}) = P(X_1, X_2, .., X_d)$$

$$p(\mathbf{X}) = p(X_1, X_2, .., X_d)$$

- **Parametric distribution models:**
  - Bernoulli (outcome of coin flip)
  - Binomial (outcome of multiple coin flips)
  - Multinomial (outcome of die)
  - Poisson
  - Exponential
  - Gamma distribution
  - Gaussian (this one is multivariate)

Modeling complex multivariate distributions

How to model/parameterize complex multivariate distributions $P(\mathbf{X})$ with a large number of variables?

One solution:

- Decompose the distribution. Reduce the number of parameters, using some form of independence.

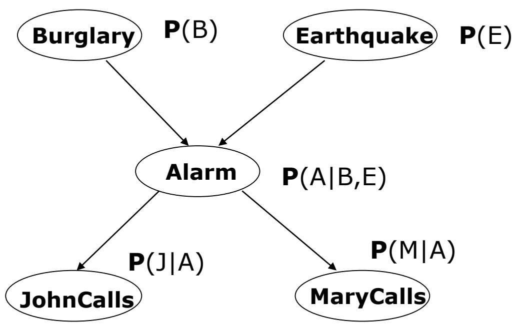
Two graphical models:

- **Bayesian belief networks (BBNs)**
- **Markov Random Fields (MRFs)**

- **Learning of these models** relies on the decomposition.

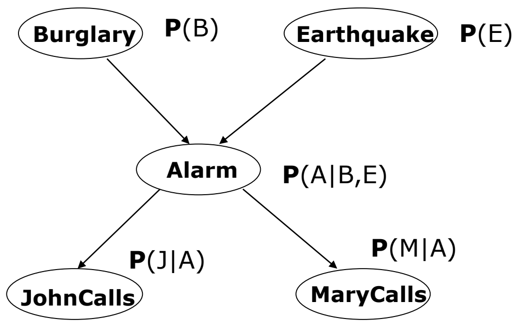
Bayesian belief network.

1. **Directed acyclic graph**
   - **Nodes** = random variables
   - **Links** = direct (causal) dependencies between variables
     - Missing links encode independences

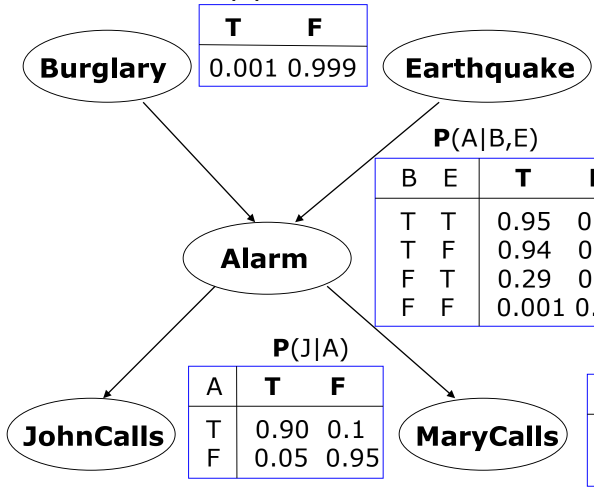


Bayesian belief network.

2. **Local conditional distributions**
   - relate variables and their parents



Bayesian belief network.



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

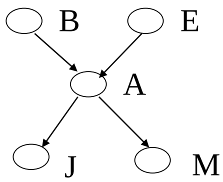
CS 3750 Advanced Machine Learning

Full joint distribution in BBNs

Full joint distribution is defined in terms of local conditional distributions (obtained via the chain rule):

$$\mathbf{P}(X_1, X_2,.., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

Example:



Assume the following assignment of values to random variables

$B = T, E = T, A = T, J = T, M = F$

Then its probability is:

$P(B = T, E = T, A = T, J = T, M = F) =$

$P(B = T)P(E = T)P(A = T \mid B = T, E = T)P(J = T \mid A = T)P(M = F \mid A = T)$

Parameter complexity problem

- In the BBN the **full joint distribution** is defined as:

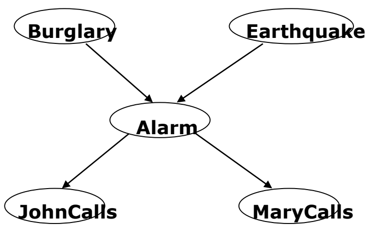
$$\mathbf{P}(X_1, X_2,.., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

Alarm example: binary (True, False) variables

# of parameters of the full joint:

?



Parameter complexity problem

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- **What did we save?**

Alarm example: binary (True, False) variables

# of parameters of the full joint:

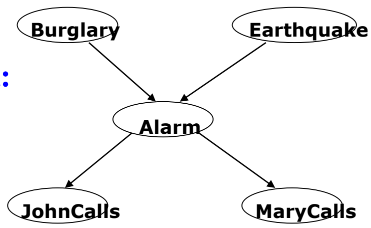
$$2^5 = 32$$

One parameter depends on the rest:

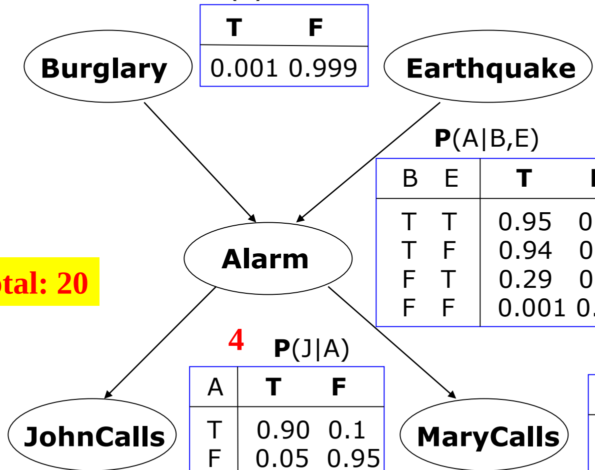
$$2^5 - 1 = 31$$

# of parameters of the BBN:

?



Bayesian belief network: parameters count



P(B) 2

T	F
0.001	0.999

P(E) 2

T	F
0.002	0.998

P(A|B,E) 8

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A) 4

A	T	F
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Parameter complexity problem

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- **What did we save?**

Alarm example: 5 binary (True, False) variables

# of parameters of the full joint:

$$2^5 = 32$$

One parameter depends on the rest:

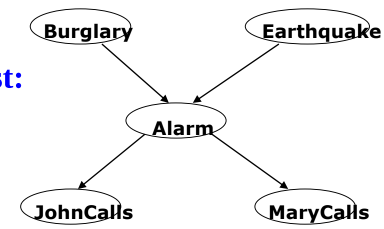
$$2^5 - 1 = 31$$

# of parameters of the BBN:

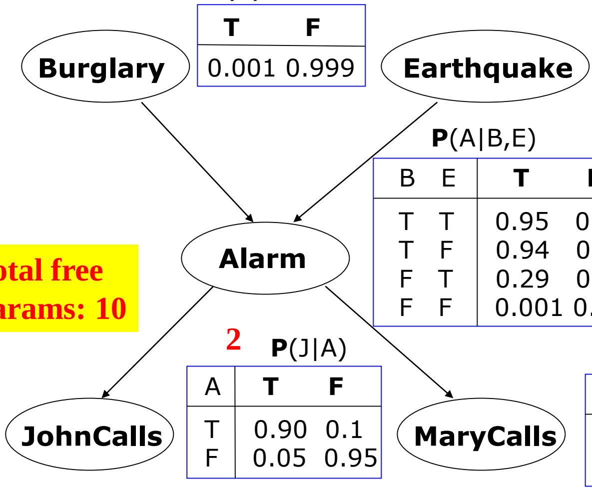
$$2^3 + 2(2^2) + 2(2) = 20$$

One parameter in every conditional depends on the rest:

?



Bayesian belief network: free parameters



P(B) 1

T	F
0.001	0.999

P(E) 1

T	F
0.002	0.998

P(A|B,E) 4

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
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P(J|A) 2

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Parameter complexity problem

- In the BBN the **full joint distribution** is defined as:

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

Alarm example: 5 binary (True, False) variables

# of parameters of the full joint:

$$2^5 = 32$$

One parameter depends on the rest:

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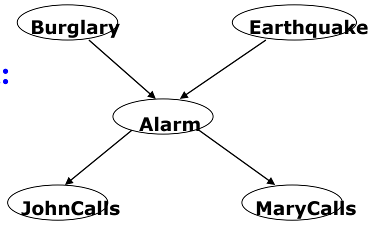
# of parameters of the BBN:

$$2^3 + 2(2^2) + 2(2) = 20$$

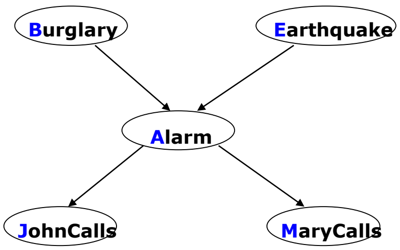


One parameter in every conditional depends on the rest:

$$2^2 + 2(2) + 2(1) = 10$$

Inference in Bayesian network

- **Bad news:**
  - Exact inference problem in BBNs is NP-hard (Cooper)
  - Approximate inference is NP-hard (Dagum, Luby)
- **But** very often we can achieve significant improvements
- Assume our Alarm network



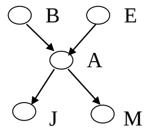
- Assume we want to compute: $P(J = T)$

Inference in Bayesian networks

- Full joint uses the decomposition
- **Calculation of marginals:**
  - Requires summation over variables we want to take out

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(B=b,E=e,A=a,J=T,M=m)$$



- How to compute sums and products more efficiently?

$$\sum_x af(x)=a\sum_x f(x)$$

Variable elimination

- **Variable elimination:**
  - E.g. Query $P(J=T)$ requires to eliminate A,B,E,M and this can be done in different order

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

Variable elimination

Assume order: **M, E, B, A** to calculate $P(J=T)$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J=T \mid A=a) P(M=m \mid A=a) P(A=a \mid B=b, E=e) P(B=b) P(E=e)$$

Variable elimination

Assume order: **M**, **E, B**,**A** to calculate $P(J=T)$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J=T \mid A=a) P(M=m \mid A=a) P(A=a \mid B=b, E=e) P(B=b) P(E=e)$$

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Variable elimination

Assume order: **M, E, B, A** to calculate $P(J=T)$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

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$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e) \quad 1$$

Variable elimination

Assume order: **M, E, B, A** to calculate $P(J=T)$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)\left[\sum_{m\in T,F} P(M=m\mid A=a)\right]$$

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Variable elimination

Assume order: M, E, B, A to calculate $P(J=T)$

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$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)\quad 1$$

$$= \sum_{a\in T,F}\sum_{b\in T,F} P(J=T\mid A=a)P(B=b)\left[\sum_{e\in T,F} P(A=a\mid B=b,E=e)P(E=e)\right]$$

$$\tau_1(A=a,B=b)$$

$\tau_1(A=a,B=b) =$

	$A=T$	$A=F$
$B=T$		
$B=F$		

$\sum_{e\in T,F} P(A=F\mid B=T,E=e)P(E=e)$

$\sum_{e\in T,F} P(A=F\mid B=F,E=e)P(E=e)$

Variable elimination

Assume order: M, E, B, A to calculate $P(J=T)$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

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$$= \sum_{a\in T,F}\sum_{b\in T,F} P(J=T\mid A=a)P(B=b)\,\tau_1(A=a,B=b)$$

Variable elimination

Assume order: **M, E, B, A** to calculate $P(J=T)$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\,|\,A=a)P(M=m\,|\,A=a)P(A=a\,|B=b,E=e)P(B=b)P(E=e)$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\,|\,A=a)P(A=a\,|B=b,E=e)P(B=b)P(E=e)\left[\sum_{m\in T,F} P(M=m\,|\,A=a)\right]$$

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Variable elimination

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$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\,|\,A=a)P(A=a\,|\,B=b,E=e)P(B=b)P(E=e)\quad 1$$

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$$= \sum_{a\in T,F}\sum_{b\in T,F} P(J=T\,|\,A=a)P(B=b)\,\tau_1(A=a,B=b)$$

$$= \sum_{a\in T,F} P(J=T\,|\,A=a)\left[\sum_{b\in T,F} P(B=b)\,\tau_1(A=a,B=b)\right]$$

$\tau_2(A=a)$

Variable elimination

Assume order: M, E, B, A to calculate $P(J=T)$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J=T \mid A=a) P(M=m \mid A=a) P(A=a \mid B=b, E=e) P(B=b) P(E=e)$$

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$$= \sum_{a \in T,F} P(J=T \mid A=a) \quad \tau_2(A=a)$$

Variable elimination

Assume order: M, E, B, A to calculate $P(J=T)$

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$$= \sum_{a \in T,F} P(J=T \mid A=a) \left[\sum_{e \in T,F} P(B=b) \, \tau_1(A=a, B=b) \right]$$

$$= \sum_{a \in T,F} P(J=T \mid A=a) \quad \tau_2(A=a) \quad = \boxed{P(J=T)}$$

Markov random fields

- **Probabilistic models with symmetric dependences.**
  - Typically models spatially varying quantities

$$P(x) \propto \prod_{c \in cl(x)} \phi_c(x_c)$$

$\phi_c(x_c)$ - A potential function (defined over factors)

- If $\phi_c(x_c)$ is strictly positive we can rewrite the definition in terms of a log-linear model :

$$P(x) = \frac{1}{Z} \exp\left(- \sum_{c \in cl(x)} E_c(x_c) \right)$$

Energy function

- Gibbs (Boltzman) distribution

$$Z = \sum_{x \in \{x\}} \exp\left(- \sum_{c \in cl(x)} E_c(x_c) \right)$$

- A partition function

Graphical representation of MRFs

An undirected network (also called independence graph)

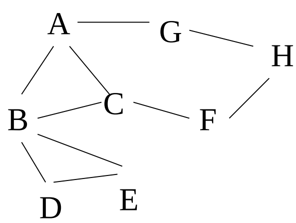
- G = (S, E)
  - S=1, 2, .. N correspond to random variables
  - $(i, j) \in E \Leftrightarrow \exists c : \{i, j\} \subset c$

    or x_i and x_j appear within the same factor c

Example:

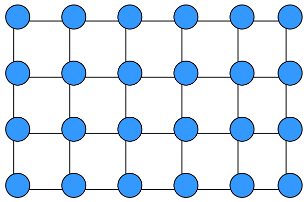
- variables A,B ..H
- Assume the full joint of MRF

$$P(A, B, \ldots H) \sim \phi_1(A, B, C)\phi_2(B, D, E)\phi_3(A, G) \phi_4(C, F)\phi_5(G, H)\phi_6(F, H)$$

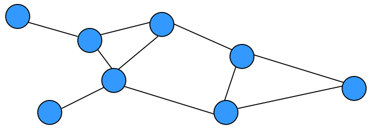


Markov random fields

- **regular lattice (Ising model)**

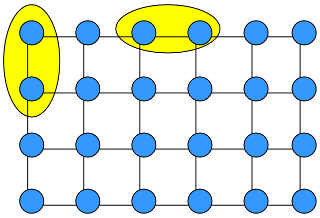


- **Arbitrary graph**

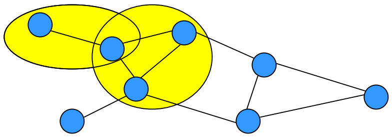


Markov random fields

- **regular lattice (Ising model)**



- **Arbitrary graph**



Markov random fields: independence relations

- **Pairwise Markov property**
  - Two nodes in the network that are not directly connected can be made independent given all other nodes
- **Local Markov property**
  - A set of nodes (variables) can be made independent from the rest of nodes variables given its immediate neighbors
- **Global Markov property**
  - A vertex set A is independent of the vertex set B (A and B are disjoint) given set C if all chains in between elements in A and B intersect C

Types of Markov random fields

- **MRFs with discrete random variables**
  - Clique potentials can be defined by mapping all clique-variable instances to R
  - Example: Assume two binary variables A,B with values {a1,a2,a3} and {b1,b2} are in the same clique c. Then:

$\phi_c(A, B) \cong$

a1	b1	0.5
a1	b2	0.2
a2	b1	0.1
a2	b2	0.3
a3	b1	0.2
a3	b2	0.4

Types of Markov random fields

- **Gaussian Markov Random Field**

$$\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

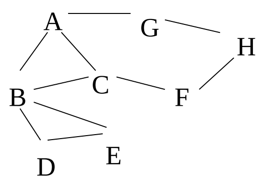
$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu}) \right]$$

- **Precision matrix** $\boldsymbol{\Sigma}^{-1}$
- **Variables in x are connected in the network only if they have a nonzero entry in the precision matrix**
  - All zero entries are not directly connected

MRF variable elimination inference

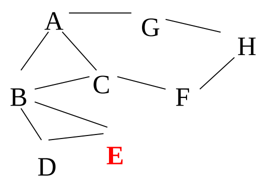
Example:

$$P(B) = \sum_{A,C,D,..H} P(A, B, ...H)$$



$$= \frac{1}{Z} \sum_{A,C,D,..H} \phi_1(A,B,C)\phi_2(B,D,E)\phi_3(A,G)\phi_4(C,F)\phi_5(G,H)\phi_6(F,H)$$

Eliminate E



$$= \frac{1}{Z} \sum_{A,C,D,F,G,H} \phi_1(A,B,C) \underbrace{\left[\sum_{E} \phi_2(B,D,E) \right]}_{\tau_1(B,D)} \phi_3(A,G)\phi_4(C,F)\phi_5(G,H)\phi_6(F,H$$

Factors

- **Factor:** is a function that maps value assignments for a subset of random variables to $\Re$ (reals)
- **The scope of the factor:**
  – a set of variables defining the factor
- **Example:**
  – Assume discrete random variables x (with values a1,a2, a3) and y (with values b1 and b2)
  – Factor:

$$\phi(x, y) \rightarrow$$

a1	b1	0.5
a1	b2	0.2
a2	b1	0.1
a2	b2	0.3
a3	b1	0.2
a3	b2	0.4

  – Scope of the factor:

$$\{x, y\}$$

Factor Product

Variables: A,B,C

$$\phi(A, B, C) = \phi(B, C) \circ \phi(A, B)$$

$\phi(B, C)$

b1	c1	0.1
b1	c2	0.6
b2	c1	0.3
b2	c2	0.4

$\phi(A, B)$

a1	b1	0.5
a1	b2	0.2
a2	b1	0.1
a2	b2	0.3
a3	b1	0.2
a3	b2	0.4

$\phi(A, B, C)$

a1	b1	c1	0.5*0.1
a1	b1	c2	0.5*0.6
a1	b2	c1	0.2*0.3
a1	b2	c2	0.2*0.4
a2	b1	c1	0.1*0.1
a2	b1	c2	0.1*0.6
a2	b2	c1	0.3*0.3
a2	b2	c2	0.3*0.4
a3	b1	c1	0.2*0.1
a3	b1	c2	0.2*0.6
a3	b2	c1	0.4*0.3
a3	b2	c2	0.4*0.4

Factor Marginalization

Variables: A,B,C

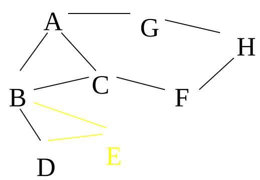
$$\phi(A,C) = \sum_{B} \phi(A,B,C)$$

a1	b1	c1	0.2
a1	b1	c2	0.35
a1	b2	c1	0.4
a1	b2	c2	0.15
a2	b1	c1	0.5
a2	b1	c2	0.1
a2	b2	c1	0.3
a2	b2	c2	0.2
a3	b1	c1	0.25
a3	b1	c2	0.45
a3	b2	c1	0.15
a3	b2	c2	0.25

a1	c1	0.2+0.4=0.6
a1	c2	0.35+0.15=0.5
a2	c1	0.8
a2	c2	0.3
a3	c1	0.4
a3	c2	0.7

MRF variable elimination inference

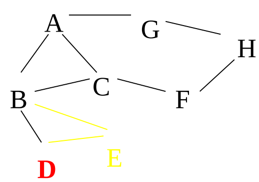
Example (cont):



$$P(B) = \sum_{A,C,D,..H} P(A,B,...H)$$

$$= \sum_{A,C,D,F,G,H} \phi_1(A,B,C)\tau_1(B,D)\phi_3(A,G)\phi_4(C,F)\phi_5(G,H)\phi_6(F,H)$$

Eliminate D

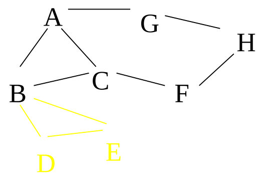


$$= \sum_{A,C,F,G,H} \phi_1(A,B,C)\underbrace{\left[\sum_{D} \tau_1(B,D)\right]}_{\tau_2(B)}\phi_3(A,G)\phi_4(C,F)\phi_5(G,H)\phi_6(F,H)$$

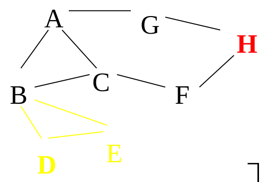
MRF variable elimination inference

Example (cont):

$$P(B) = \sum_{A,C,D,..H} P(A,B,...H)$$



$$= \sum_{A,C,F,G,H} \phi_1(A,B,C)\tau_2(B)\phi_3(A,G)\phi_4(C,F)\phi_5(G,H)\phi_6(F,H)$$



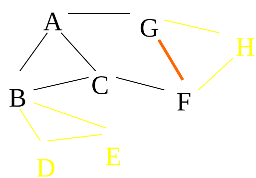
Eliminate H

$$= \sum_{A,C,F,G} \phi_1(A,B,C)\tau_2(B)\phi_3(A,G)\phi_4(C,F)\left[\underbrace{\sum_{H} \underbrace{\phi_5(G,H)\phi_6(F,H)}_{\tau_3(F,G,H)}}_{\tau_4(F,G)}\right]$$

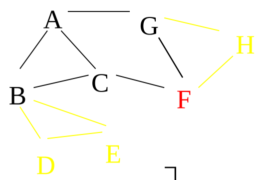
MRF variable elimination inference

Example (cont):

$$P(B) = \sum_{A,C,D,..H} P(A,B,...H)$$



$$= \sum_{A,C,F,G} \phi_1(A,B,C)\tau_2(B)\phi_3(A,G)\phi_4(C,F)\tau_4(F,G)$$

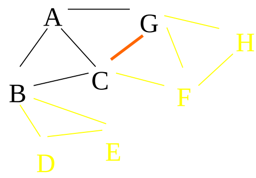


Eliminate F

$$= \sum_{A,C,G} \phi_1(A,B,C)\tau_2(B)\phi_3(A,G)\left[\underbrace{\sum_{F} \underbrace{\phi_4(C,F)\tau_4(F,G)}_{\tau_5(C,F,G)}}_{\tau_6(G,C)}\right]$$

MRF variable elimination inference

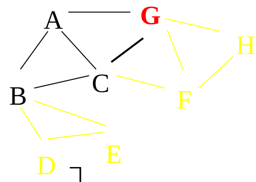
Example (cont):



$$P(B) = \sum_{A,C,D,..H} P(A,B,...H)$$

$$= \sum_{A,C,G} \phi_1(A,B,C)\tau_2(B)\phi_3(A,G)\tau_6(C,G)$$

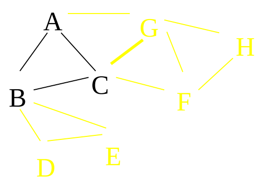
Eliminate G



$$= \underbrace{\sum_{A,C} \phi_1(A,B,C)\tau_2(B)\left[\sum_F \underbrace{\phi_3(A,G)\tau_6(C,G)}_{\tau_7(A,C,G)}\right]}_{\tau_8(A,C)}$$

MRF variable elimination inference

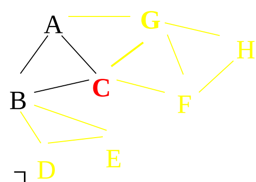
Example (cont):



$$P(B) = \sum_{A,C,D,..H} P(A,B,...H)$$

$$= \sum_{A,C} \phi_1(A,B,C)\tau_2(B)\tau_8(A,C)$$

Eliminate C



$$= \underbrace{\sum_{A} \tau_2(B)\left[\sum_C \underbrace{\phi_1(A,B,C)\tau_8(A,C)}_{\tau_9(A,B,C)}\right]}_{\tau_{10}(A,B)}$$

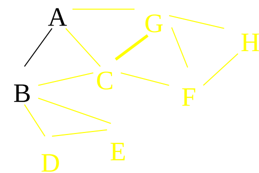
MRF variable elimination inference

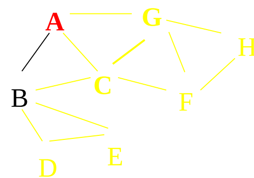
Example (cont):

$$P(B) = \sum_{A,C,D,..H} P(A,B,...H)$$

$$= \sum_A \tau_2(B)\tau_{10}(A,B)$$

$$= \tau_2(B)\sum_A \tau_{10}(A,B)$$

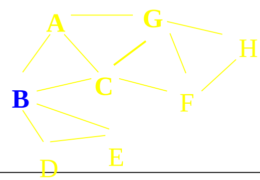




Eliminate A

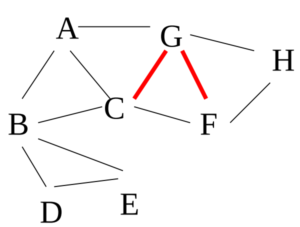
$$= \tau_2(B)\underbrace{\sum_A \tau_{10}(A,B)}_{\tau_{11}(B)}$$

$$= \tau_2(B)\tau_{11}(B)$$



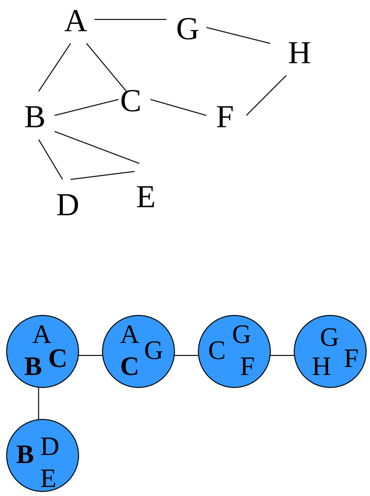
Induced graph

- **A graph induced by a specific variable elimination order:**
- **a graph G extended by links that represent intermediate factors**
  - .



Tree decomposition of the graph

- **A tree decomposition of a graph G:**
  - A tree *T* with a vertex set associated to every node.
  - For all edges $\{v,w\} \in G$: there is a set containing both *v* and *w* in *T*.
  - For every $v \in G$: the nodes in T that contain *v* form a connected subtree.



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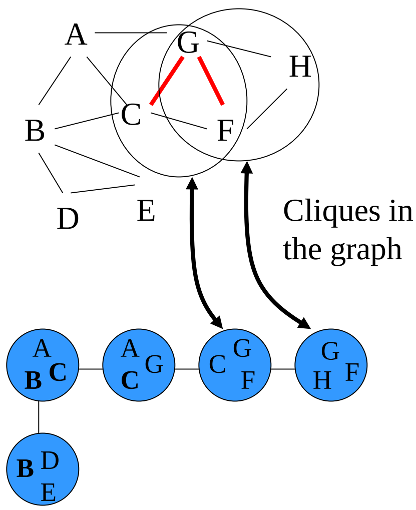

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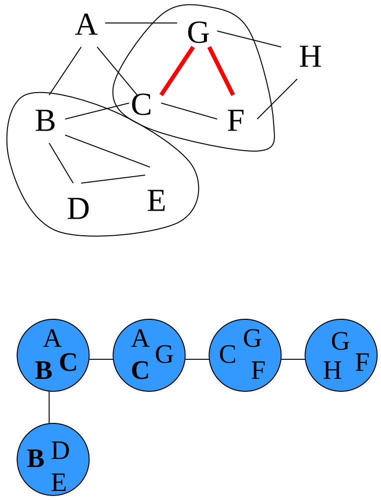
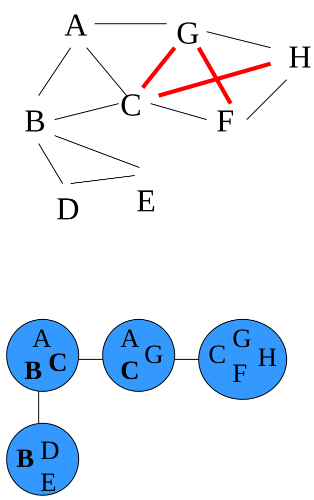


CS 3750 Advanced Machine Learning

Tree decomposition of the graph

- **Another tree decomposition of a graph G:**
  - A tree *T* with a vertex set associated to every node.
  - For all edges $\{v,w\} \in G$: there is a set containing both *v* and *w* in *T*.
  - For every $v \in G$: the nodes in T that contain *v* form a connected subtree.



CS 3750 Advanced Machine Learning

Tree decomposition of the graph

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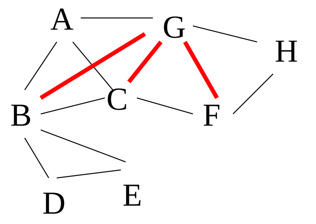
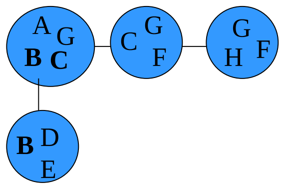


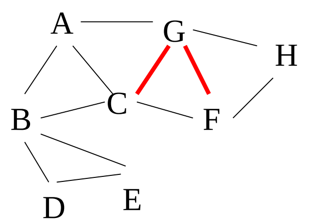


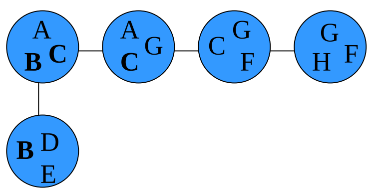
Treewidth of the graph

- **Width** of the tree decomposition:

  $$\max_{i \in I} |X_i| - 1$$

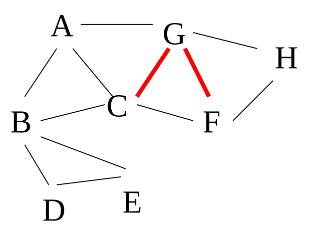
- **Treewidth** of a graph *G*: tw(*G*)= minimum width over all tree decompositions of *G*.

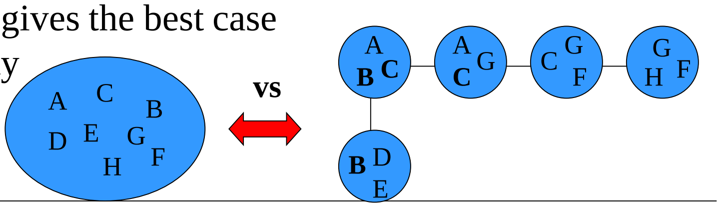




Treewidth of the graph

- **Treewidth** of a graph *G*: tw(*G*)= minimum width over all tree decompositions of *G*
- Why is it important?
- The calculations can take advantage of the structure and be performed more efficiently
- treewidth gives the best case complexity

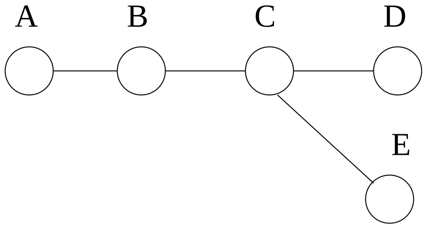




Trees

Why do we like trees?

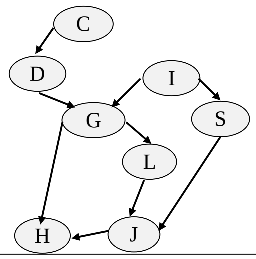
- Inference in trees structures can be done in time **linear in the number of nodes in the tree**

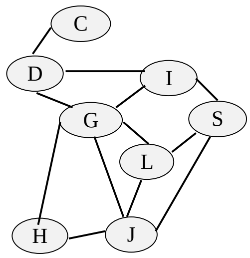


Converting BBNs to MRFs

Moral-graph H[G]: of a Bayesian network over X is an undirected graph over X that contains an edge between x and y if:

- There exists a directed edge between them in G.
- They are both parents of the same node in G.





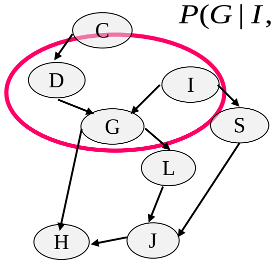
Moral Graphs

Why moralization?

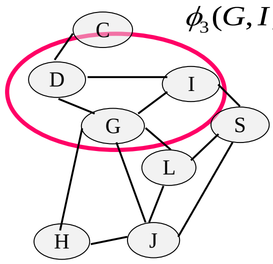
$$P(C,D,G,I,S,L,J,H) =$$

$$= P(C)P(D|C)P(G|I,D)P(S|I)P(L|G)P(J|L,S)P(H|G,J)$$

$$= \phi_1(C)\phi_2(D,C)\phi_3(G,I,D)\phi_4(S,I)\phi_5(L,G)\phi_6(J,L,S)\phi_7(H,G,J)$$



$P(G|I,D)$

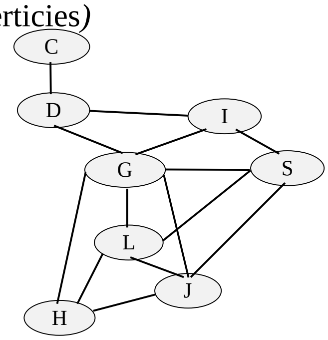


$\phi_3(G,I,D)$

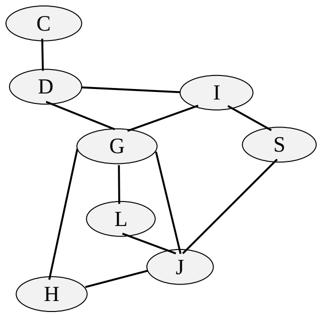
Chordal graphs

Chordal Graph: an undirected graph *G*

- all cycles of four or more vertices have a chord (another edge breaking the cycle)
- minimum cycle for every vertex in a cycle is 3 (contains 3 verticies)



Chordal.

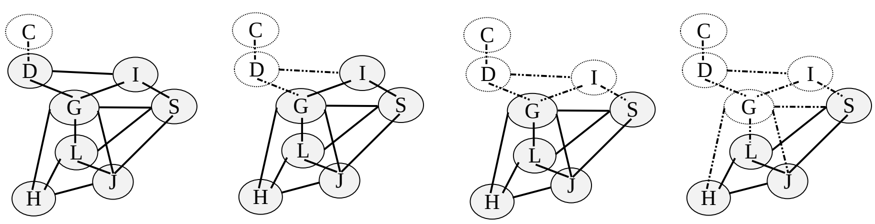


Not Chordal

Chordal Graphs

Properties:

- There exists an elimination ordering that adds no edges.
- The minimal induced treewidth of the graph is equal to the size of the largest clique - 1.



Triangulation

The process of converting a graph G into a chordal graph is called Triangulation.

A new graph obtained via triangulation is:

1) Guaranteed to be chordal.
2) Not guaranteed to be (treewidth) optimal.

- There exist exact algorithms for finding the **minimal chordal graphs**, and heuristic methods with a guaranteed upper bound.

Chordal Graphs

- Given a minimum triangulation for a graph *G*, we can carry out the variable-elimination algorithm in the minimum possible time.

- **Complexity** of the optimal triangulation:
  - Finding the minimal triangulation is **NP-Hard**.

- **The inference limit:**
  - Inference time is exponential in terms of the largest clique (factor) in *G*.

Conclusions: MRFs

- We cannot escape **exponential costs in the treewidth**
- But in many graphs the tree-width is much smaller than the total number of variables
- Still a problem: Finding the optimal decomposition is hard
  - But, paying the cost up front may be worth it.
  - Triangulate once, query many times.
  - Real cost savings if not a bounded one