CS 3710 Advanced Topics in AI
Lecture 18

Density estimation

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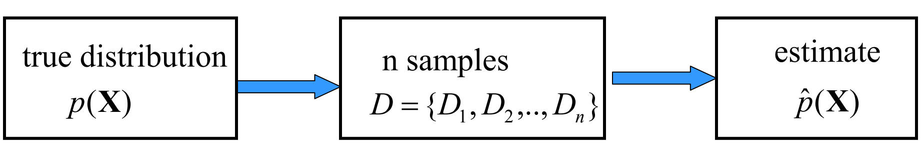
Density estimation

Data: $D = \{D_1, D_2, .., D_n\}$

$D_i = \mathbf{x}_i$ a vector of attribute values

Objective: try to estimate the underlying true probability distribution over variables $\mathbf{X}$, $p(\mathbf{X})$, using examples in D



Standard (iid) assumptions: Samples
- **are independent of each other**
- **come from the same (identical) distribution (fixed $p(\mathbf{X})$)**

Density estimation

Types of density estimation:

Parametric

- the distribution is modeled using a set of parameters Θ

$$p(\mathbf{X} \mid \Theta)$$

- **Example:** mean and covariances of multivariate normal
- **Estimation:** find parameters $\hat{\Theta}$ that fit the data D the best

Non-parametric

- The model of the distribution utilizes all examples in D
- As if all examples were parameters of the distribution
- **Examples:** Nearest-neighbor

Semi-parametric

Parametric density estimation

P**arametric density estimation**

Basic settings:

- A set of random variables $\mathbf{X} = \{X_1, X_2, \ldots, X_d\}$
- **A model of the distribution** over variables in ***X*** with parameters Θ
- **Data** $D = \{D_1, D_2, .., D_n\}$

Objective: find parameters $\hat{\Theta}$ that describe $p(\mathbf{X} \mid \Theta)$ the best

Parameter learning

What is the best set of parameters?

- **Maximum likelihood (ML) estimates**

  maximize $p(D \mid \Theta, \xi)$

  ξ - represents prior (background) knowledge

- **Maximum a posteriori probability (MAP) estimate**

  maximize $p(\Theta \mid D, \xi)$

  Selects the mode of the posterior

$$p(\Theta \mid D, \xi) = \frac{p(D \mid \Theta, \xi)\, p(\Theta \mid \xi)}{p(D \mid \xi)}$$

Parameter learning

- **Both ML or MAP pick one parameter value**
  – Is it always the best solution?
- **Bayesian approach**
  – Remedies the limitation of one choice
  – Keeps and uses complete posterior distribution $p(\Theta \mid D, \xi)$
  – Optimization is replaced with integration

- **How is it used? Assume we want:** $P(\mathbf{x} \mid D, \xi)$
  – Consider all parameter settings and averages the result

$$P(\mathbf{x} \mid D, \xi) = \int_{\theta} P(\mathbf{x} \mid \theta, \xi)\, p(\theta \mid D, \xi)\, d\theta$$

  – **Example:** predict the result of the outcome x=1

$$P(x = 1 \mid D, \xi)$$

Binomial distribution.

Data: D a set of order-independent outcomes with two possible values – 0 or 1 (head or tail)

N_1 - number of heads seen N_2 - number of tails seen

We treat *D* as a multi-set !!!

Model: probability of a 1 θ
probability of a 0 $(1-\theta)$

Probability of an outcome

$$P(D \mid \theta) = \binom{N}{N_1} \theta^{N_1} (1-\theta)^{N_2}$$ **Binomial distribution**

Maximum likelihood (ML) estimate.

Likelihood of data:

$$P(D \mid \theta) = \binom{N}{N_1} \theta^{N_1} (1-\theta)^{N_2} = \frac{N!}{N_1! N_2!} \theta^{N_1} (1-\theta)^{N_2}$$

Log-likelihood

$$l(D,\theta) = \log \binom{N}{N_1} \theta^{N_1} (1-\theta)^{N_2} = \log \frac{N!}{N_1! N_2!} + N_1 \log\theta + N_2 \log(1-\theta)$$

Constant from the point of optimization !!!

ML Solution: $\theta_{ML} = \frac{N_1}{N} = \frac{N_1}{N_1 + N_2}$

The same as for Bernoulli and D **with iid** sequence of examples

Maximum a posteriori estimate

MAP estimate $\theta_{MAP} = \arg\max_{\theta} p(\theta \mid D, \xi)$

$$p(\theta \mid D, \xi) = \frac{P(D \mid \theta, \xi) p(\theta \mid \xi)}{P(D \mid \xi)}$$ **(via Bayes rule)**

Prior choice

$$p(\theta \mid \xi) = Beta(\theta \mid \alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \theta^{\alpha_1 - 1}(1-\theta)^{\alpha_2 - 1}$$

Likelihood

$$P(D \mid \theta) = \frac{\Gamma(N_1 + N_2)}{\Gamma(N_1)\Gamma(N_2)} \theta^{N_1}(1-\theta)^{N_2}$$

Posterior

$$p(\theta \mid D, \xi) = Beta(\alpha_1 + N_1, \alpha_2 + N_2)$$

MAP estimate

$$\theta_{MAP} = \frac{\alpha_1 + N_1 - 1}{\alpha_1 + \alpha_2 + N_1 + N_2 - 2}$$

Bayesian learning, expectation

The result is the same as for Bernoulli distribution

$$E(\theta) = \int_0^1 \theta Beta(\theta \mid \eta_1, \eta_2) d\theta = \frac{\eta_1}{\eta_1 + \eta_2}$$

Expected value of the parameter

$$E(\theta) = \frac{\alpha_1 + N_1}{\alpha_1 + N_1 + \alpha_2 + N_2}$$

Predictive probability of an event **x**=1

$$E(\theta) = P(x = 1 \mid \theta, \xi) = \frac{\alpha_1 + N_1}{\alpha_1 + N_1 + \alpha_2 + N_2}$$

Multinomial distribution

A kind of multi-way coin toss (roll of dice)

- **Data:** a set of N outcomes (multi-set)

  N_i - a number of times an outcome i has been seen

Model parameters: $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots \theta_k)$ **s.t.** $\sum_{i=1}^{k} \theta_i = 1$

θ_i - probability of an outcome i

Probability of data (likelihood)

$$P(N_1, N_2, \ldots N_k \mid \boldsymbol{\theta}, \xi) = \frac{N!}{N_1! N_2! \ldots N_k!} \theta_1^{N_1} \theta_2^{N_2} \ldots \theta_k^{N_k}$$

Multinomial distribution

ML estimate:

$$\theta_{i,ML} = \frac{N_i}{N}$$

MAP estimate

Choice of prior: Dirichlet distribution

$$Dir(\boldsymbol{\theta} \mid \alpha_1, .., \alpha_k) = \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \theta_1^{\alpha_1 - 1} \theta_2^{\alpha_2 - 1} \ldots \theta_k^{\alpha_k - 1}$$

Dirichlet is the conjugate choice for multinomial

$$P(D \mid \boldsymbol{\theta}, \xi) = P(N_1, N_2, \ldots N_k \mid \boldsymbol{\theta}, \xi) = \frac{N!}{N_1! N_2! \ldots N_k!} \theta_1^{N_1} \theta_2^{N_2} \ldots \theta_k^{N_k}$$

Posterior distribution

$$p(\boldsymbol{\theta} \mid D, \xi) = \frac{P(D \mid \boldsymbol{\theta}, \xi) Dir(\boldsymbol{\theta} \mid \alpha_1, \alpha_2, .. \alpha_k)}{P(D \mid \xi)} = Dir(\boldsymbol{\theta} \mid \alpha_1 + N_1, .., \alpha_k + N_k)$$

MAP estimate:

$$\theta_{i,MAP} = \frac{\alpha_i + N_i - 1}{\sum_{i=1,..k} (\alpha_i + N_i) - k}$$

Bayesian learning

The result is analogous to the result for binomial

$$E(\boldsymbol{\theta}) = \oint_{0\le\theta_i\le 1, \sum\theta_i=1} \boldsymbol{\theta} \; Dir(\boldsymbol{\theta} \mid \boldsymbol{\eta}) d\boldsymbol{\theta} = \left(\frac{\eta_1}{\eta_1+\eta_2+\eta_k}, \ldots \frac{\eta_i}{\eta_1+\eta_2+\eta_k}, \ldots \frac{\eta_k}{\eta_1+\eta_2+\eta_k} \right)$$

Bayesian estimate substitutes posterior

$$E(\boldsymbol{\theta}) = \left(\frac{\alpha_1+N_1}{\alpha_1+N_1+\ldots+\alpha_k+N_k} \ldots \frac{\alpha_i+N_i}{\alpha_1+N_1+\ldots+\alpha_k+N_k} \ldots \frac{\alpha_k+N_k}{\alpha_1+N_1+\ldots+\alpha_k+N_k} \right)$$

Represents the predictive probability of an event x=i

$$P(x=i \mid \boldsymbol{\theta}, \xi) = \frac{\alpha_i+N_i}{\alpha_1+N_1+\ldots+\alpha_k+N_k}$$

Other distributions

The same ideas can be applied to other distributions

– Typically we choose distributions that behave in a nice way so that the computations lead to "nice" solutions

- **Exponential family of distributions**

Conjugate choices for some of the distributions from the exponential family:

– **Binomial – Beta**
– **Multinomial - Dirichlet**
– **Exponential – Gamma**
– **Poisson - Gamma**
– **Gaussian - Gaussian (mean) and Wishart (covariance)**

Distributions from the exponential family

Gamma distribution:

$$p(x \mid a,b) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-\frac{x}{b}} \qquad \text{for} \quad x \in [0,\infty]$$

Exponential distribution:

- A special case of Gamma for a=1

$$p(x \mid b) = \left(\frac{1}{b}\right) e^{-\frac{x}{b}}$$

Poisson distribution:

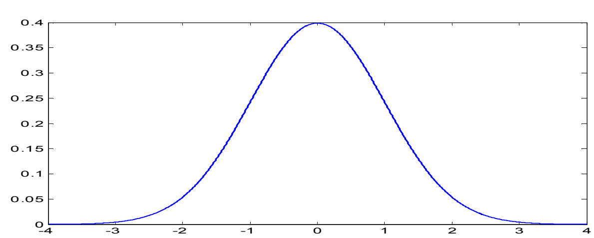
$$p(x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{for} \quad x \in \{0,1,2,\ldots\}$$

Gaussian (normal) distribution

- **Gaussian:** $x \sim N(\mu,\sigma)$
- **Parameters:** μ - mean
  σ - standard deviation
- **Density function:**

$$p(x \mid \mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2]$$

- **Example:**



$N(0,1)$

Parameter estimates

- **Loglikelihood** $$l(D, \mu, \sigma) = \log \prod_{i=1}^{n} p(x_i \mid \mu, \sigma)$$
- **ML estimates of the mean and variance:**

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

  – ML variance estimate is biased

$$E_n(\sigma^2) = E_n\left(\frac{1}{n}\sum_{i=1}^{n} (x_i - \hat{\mu})^2\right) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

- **Unbiased estimate:**

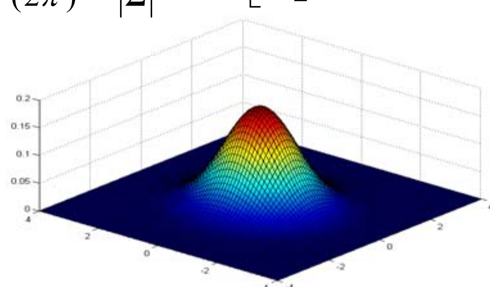
$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

Multivariate normal distribution

- **Multivariate normal:** $\mathbf{x} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- **Parameters:** $\boldsymbol{\mu}$ - mean
  $\boldsymbol{\Sigma}$ - covariance matrix
- **Density function:**

$$p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right]$$

- **Example:**



Parameter estimates

- **Loglikelihood** $l(D, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \log \prod_{i=1}^{n} p(\mathbf{x}_i \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})$
- **ML estimates of the mean and covariances:**

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T$$

  – Covariance estimate is biased

$$E_n(\hat{\boldsymbol{\Sigma}}) = E_n\left(\frac{1}{n}\sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T\right) = \frac{n-1}{n}\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}$$

- **Unbiased estimate:**

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{i=1}^{n} (\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})^T$$

Posterior of a multivariate normal

- **Assume a prior on the mean $\boldsymbol{\mu}$ that is normally distributed:**

$$p(\boldsymbol{\mu}) \approx N(\boldsymbol{\mu}_p, \boldsymbol{\Sigma}_p)$$

- **Then the posterior of $\boldsymbol{\mu}$ is normally distributed**

$$p(\boldsymbol{\mu} \mid D) \approx \left(\prod_{i=1}^{n} \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}_i - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}_i - \boldsymbol{\mu})\right]\right)$$

$$* \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}_p|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{\mu}_p)^T \boldsymbol{\Sigma}_p^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_p)\right]$$

$$= \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}_n|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{\mu}_n)^T \boldsymbol{\Sigma}_n^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_n)\right]$$

Posterior of a multivariate normal

- **Then the posterior of $\boldsymbol{\mu}$ is normally distributed**

$$p(\boldsymbol{\mu} \mid D) = \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}_n|^{1/2}} \exp\left[-\frac{1}{2} (\boldsymbol{\mu} - \boldsymbol{\mu}_n)^T \boldsymbol{\Sigma}_n^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_n) \right]$$

$$\boldsymbol{\Sigma}_n^{-1} = n\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Sigma}_p^{-1}$$

$$\boldsymbol{\mu}_n = \boldsymbol{\Sigma}_p \left(\boldsymbol{\Sigma}_p + \frac{1}{n} \boldsymbol{\Sigma} \right)^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right) \quad + \frac{1}{n} \boldsymbol{\Sigma} \left(\boldsymbol{\Sigma}_p + \frac{1}{n} \boldsymbol{\Sigma} \right)^{-1} \boldsymbol{\mu}_p$$

$$\boldsymbol{\Sigma}_n = \boldsymbol{\Sigma}_p \left(\boldsymbol{\Sigma}_p + \frac{1}{n} \boldsymbol{\Sigma} \right)^{-1} \frac{1}{n} \boldsymbol{\Sigma}$$

Recursive Bayesian parameter estimation.

- **Recursive Bayesian approach**
  - Estimates of the posterior can be sometimes computed incrementally for a sequence of data points

$$p(\Theta \mid D, \xi) = \frac{p(D \mid \Theta, \xi)\, p(\Theta \mid \xi)}{\int_{\Theta} p(D \mid \Theta, \xi)\, p(\Theta \mid \xi)\, d\Theta}$$

- If we use a conjugate prior we get back the same posterior
- Assume we split the data D in the last element **x** and the rest

$$p(D \mid \boldsymbol{\Theta}) = P(x \mid \boldsymbol{\Theta}) P(D_{n-1} \mid \boldsymbol{\Theta})$$

- **Then:**

$$p(\Theta \mid D, \xi) = \frac{P(x \mid \boldsymbol{\Theta}) \overbrace{P(D_{n-1} \mid \boldsymbol{\Theta})\, p(\Theta \mid \xi)}^{\text{A “new” prior}}}{\int_{\Theta} P(x \mid \boldsymbol{\Theta}) P(D_{n-1} \mid \boldsymbol{\Theta})\, p(\Theta \mid \xi)\, d\Theta}$$

Exponential family

Exponential family:

- all probability mass / density functions that can be written in the exponential normal form

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x})\right]$$

- $\boldsymbol{\eta}$ a vector of natural (or canonical) parameters
- $t(\mathbf{x})$ a function referred to as a sufficient statistic
- $h(\mathbf{x})$ a function of x (it is less important)
- $Z(\boldsymbol{\eta})$ a normalization constant

$$Z(\boldsymbol{\eta}) = \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}$$

- Other common form:

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x}) - A(\boldsymbol{\eta})\right] \qquad \log Z(\boldsymbol{\eta}) = A(\boldsymbol{\eta})$$

Exponential family: examples

- **Bernoulli distribution**

$$p(x \mid \pi) = \pi^x (1-\pi)^{1-x}$$

$$= \exp\left\{\log\left(\frac{\pi}{1-\pi}\right)x + \log(1-\pi)\right\}$$

$$= \exp\{\log(1-\pi)\}\exp\left\{\log\left(\frac{\pi}{1-\pi}\right)x\right\}$$

- **Exponential family**

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x})\right]$$

- **Parameters**

$\boldsymbol{\eta} = ?$ $\qquad$ $t(\mathbf{x}) = ?$

$Z(\boldsymbol{\eta}) = ?$ $\qquad$ $h(\mathbf{x}) = ?$

Exponential family: examples

- **Bernoulli distribution**

$$p(x \mid \pi) = \pi^{x}(1-\pi)^{1-x}$$
$$= \exp\left\{\log\left(\frac{\pi}{1-\pi}\right)x + \log(1-\pi)\right\}$$
$$= \exp\{\log(1-\pi)\}\exp\left\{\log\left(\frac{\pi}{1-\pi}\right)x\right\}$$

- **Exponential family**

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(\mathbf{x}) \exp\left[\boldsymbol{\eta}^{T} t(\mathbf{x})\right]$$

- **Parameters**

$$\boldsymbol{\eta} = \log\frac{\pi}{1-\pi} \quad \left(\text{note} \quad \pi = \frac{1}{1+e^{-\eta}}\right) \qquad t(\mathbf{x}) = x$$

$$Z(\boldsymbol{\eta}) = \frac{1}{1-\pi} = 1 + e^{\eta} \qquad h(\mathbf{x}) = 1$$

Exponential family: examples

- **Univariate Gaussian distribution**

$$p(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^{2}}(x-\mu)^{2}\right]$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mu^{2}}{2\sigma^{2}} - \log\sigma\right) \exp\left\{\frac{\mu}{\sigma^{2}}x - \frac{1}{2\sigma^{2}}x^{2}\right\}$$

- **Exponential family**

$$f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(x) \exp\left[\eta^{T} t(x)\right]$$

- **Parameters**

$$\boldsymbol{\eta} = ? \qquad t(\mathbf{x}) = ?$$

$$Z(\boldsymbol{\eta}) = ? \qquad h(\mathbf{x}) = ?$$

Exponential family: examples

- **Univariate Gaussian distribution**

$$p(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2} - \log\sigma\right) \exp\left\{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2\right\}$$

- **Exponential family** $f(\mathbf{x} \mid \boldsymbol{\eta}) = \frac{1}{Z(\boldsymbol{\eta})} h(x) \exp\left[\eta^T t(x)\right]$

- **Parameters**

$$\boldsymbol{\eta} = \begin{bmatrix} \mu / \sigma^2 \\ -1/2\sigma^2 \end{bmatrix} \qquad t(\mathbf{x}) = \begin{bmatrix} x \\ x^2 \end{bmatrix}$$

$$Z(\boldsymbol{\eta}) = \exp\left\{\frac{\mu^2}{2\sigma^2} + \log\sigma\right\} = \exp\left\{-\frac{\eta_1^2}{4\eta_2} - \frac{1}{2}\log(-2\eta_2)\right\}$$

$$h(\mathbf{x}) = 1/\sqrt{2\pi}$$

Exponential family

- **For iid samples, the likelihood of data is**

$$P(D \mid \boldsymbol{\eta}) = \prod_{i=1}^{n} p(\mathbf{x}_i \mid \boldsymbol{\eta}) = \prod_{i=1}^{n} h(\mathbf{x}_i) \exp\left[\boldsymbol{\eta}^T t(\mathbf{x}_i) - A(\boldsymbol{\eta})\right]$$

$$= \left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right] \exp\left[\sum_{i=1}^{n} \boldsymbol{\eta}^T t(\mathbf{x}_i) - A(\boldsymbol{\eta})\right]$$

$$= \left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right] \exp\left[\boldsymbol{\eta}^T \left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - nA(\boldsymbol{\eta})\right]$$

- **Important:**
  - the dimensionality of the sufficient statistic remains the same with the number of samples

Exponential family

- **log likelihood of data is**

$$l(D,\boldsymbol{\eta}) = \log\left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right]\exp\left[\boldsymbol{\eta}^T\left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - nA(\boldsymbol{\eta})\right]$$

$$= \log\left[\prod_{i=1}^{n} h(\mathbf{x}_i)\right] + \left[\boldsymbol{\eta}^T\left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - nA(\boldsymbol{\eta})\right]$$

- **Optimizing the loglikelihood**

$$\nabla_{\boldsymbol{\eta}} l(D,\boldsymbol{\eta}) = \left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right) - n\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) \quad = \mathbf{0}$$

- **For the ML estimate it must hold**

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \frac{1}{n}\left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right)$$

Exponential family

- **Rewritting the gradient:**

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \nabla_{\boldsymbol{\eta}} \log Z(\boldsymbol{\eta}) = \nabla_{\boldsymbol{\eta}} \log \int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}$$

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \frac{\int t(\mathbf{x}) h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}}{\int h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x})\right\} d\mathbf{x}}$$

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = \int t(\mathbf{x}) h(\mathbf{x}) \exp\left\{\boldsymbol{\eta}^T t(\mathbf{x}) - A(\boldsymbol{\eta})\right\} d\mathbf{x}$$

$$\nabla_{\boldsymbol{\eta}} A(\boldsymbol{\eta}) = E(t(\mathbf{x}))$$

- **Result:** $$E(t(\mathbf{x})) = \frac{1}{n}\left(\sum_{i=1}^{n} t(\mathbf{x}_i)\right)$$

- **For the ML estimate the parameters $\boldsymbol{\eta}$ should be adjusted such that the expectation of the statistic t(x) is equal to the observed sample statistics**

Moments of the distribution

- **For the exponential family**
  - The k-th moment of the statistic corresponds to the k-th derivative of $A(\boldsymbol{\eta})$
  - If x is a component of t(x) then we get the moments of the distribution by differentiating its corresponding natural parameter
- **Example: Bernoulli** $p(x \mid \pi) = \exp\left\{ \log\left(\frac{\pi}{1-\pi}\right) x + \log(1-\pi) \right\}$

$$A(\boldsymbol{\eta}) = \log \frac{1}{1-\pi} = \log(1+e^{\eta})$$

- **Derivatives:**

$$\frac{\partial A(\boldsymbol{\eta})}{\partial \eta} = \frac{\partial}{\partial \eta} \log(1+e^{\eta}) = \frac{e^{\eta}}{(1+e^{\eta})} = \frac{1}{(1+e^{-\eta})} = \pi$$

$$\frac{\partial A(\boldsymbol{\eta})}{\partial \eta^2} = \frac{\partial}{\partial \eta} \frac{1}{(1+e^{-\eta})} = \pi(1-\pi)$$