CS 2740 Knowledge Representation
Lecture 4

Propositional logic

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Administration

- **Homework assignment 1 is out**
  - **Due next week on Wednesday, September 17**
  - **Problems:**
    - LISP programming
    - a PL translation exercise
- **Submission guidelines:**
  - Reports should be submitted at the beginning of the class on the due date
  - Programs should be submitted electronically before the class. Submission instructions are in:
    http://www.cs.pitt.edu/~milos/courses/cs2740/program-submissions.html

Knowledge representation

- The objective of knowledge representation is to express knowledge about the world in a computer-tractable form

- Key aspects of knowledge representation languages:
  - **Syntax:** describes how sentences are formed in the language
  - **Semantics:** describes the meaning of sentences, what is it the sentence refers to in the real world
  - **Computational aspect:** describes how sentences and objects are manipulated in concordance with semantical conventions

Many KB systems rely on some variant of logic

Logic

- **Logic:**
  - defines a formal language for logical reasoning

- It gives us a tool that helps us to understand how to construct a valid argument

- **Logic Defines:**
  - the meaning of statements
  - the rules of logical inference

Logic

Logic is defined by**:**

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with values.
- **The valuation (meaning) function** V
  - Assigns a value (typically the truth value) to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

Propositional logic

- **The simplest logic**

- **Definition:**
  - **A proposition is a statement that is either true or false.**

- **Examples:**
  - Pitt is located in the Oakland section of Pittsburgh.
    - **(T)**
  - $5 + 2 = 8.$
    - **(F)**
  - It is raining today.
    - **(either T or F)**

Propositional logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - x + 5 = 3
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **(T)**
  - She is very talented.
    - **since she is not specified, neither true nor false**
  - There are other life forms on other planets in the universe.
    - **either T or F**

Propositional logic. Syntax

- **Formally propositional logic P:**
  - is defined **by Syntax+interpretation+semantics of P**

Syntax:

- **Symbols (alphabet)** in P:
  - **Constants**: ***True, False***
  - **Propositional symbols**

    Examples:
    - *P*
    - *Pitt is located in the Oakland section of Pittsburgh.,*
    - *It rains outside,* etc.
  - **A set of connectives**:

$$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$$

Propositional logic. Syntax

Sentences in the propositional logic:

- **Atomic sentences:**
  - **Constructed from constants and propositional symbols**
  - True, False are (atomic) sentences
  - P, Q or *Light in the room is on, It rains outside* are (atomic) sentences
- **Composite sentences:**
  - **Constructed from valid sentences via connectives**
  - If A, B are sentences then
    $\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \Rightarrow B) \quad (A \Leftrightarrow B)$
    or $(A \vee B) \wedge (A \vee \neg B)$
    are sentences

Propositional logic. Semantics.

The semantic gives the meaning to sentences.

the semantics in the propositional logic is defined by:

1. **Interpretation of propositional symbols and constants**
   - Semantics of atomic sentences
2. **Through the meaning of connectives**
   - Meaning (semantics) of composite sentences

Semantic: propositional symbols

A **propositional symbol**

- a statement about the world that is either true or false

  Examples:
  - *Pitt is located in the Oakland section of Pittsburgh*
  - *It rains outside*
  - *Light in the room is on*
- An **interpretation** maps symbols to one of the two values: ***True (T),*** or ***False (F)***, depending on whether the symbol is satisfied in the world

  I: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

  I': *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***

Semantic: propositional symbols

The **meaning (value)** of the propositional symbol for a specific interpretation is given by its interpretation

I: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

V(Light in the room is on, **I**) = ***True***

V(It rains outside, **I**) = ***False***

I': *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***

V(Light in the room is on, **I'**) = ***False***

Semantics: constants

- **The meaning (truth) of constants:**
  - True and False constants are always (under any interpretation) assigned the corresponding ***True,False*** value

$V(True, \mathbf{I}) = \textbf{\textit{True}}$

$V(False, \mathbf{I}) = \textbf{\textit{False}}$

For any interpretation **I**

Semantics: composite sentences

- **The meaning (truth value) of complex propositional sentences.**
  - Determined using the standard rules of logic:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
True	***True***	***False***	***True***	***True***	***True***	***True***
True	***False***	***False***	***False***	***True***	***False***	***False***
False	***True***	***True***	***False***	***True***	***True***	***False***
False	***False***	***True***	***False***	***False***	***True***	***True***

Rows define all possible interpretations (worlds)

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday. $\neg p \wedge q$
- We will go swimming only if it is sunny. $r \rightarrow p$
- If we do not go swimming then we will take a canoe trip. $\neg r \rightarrow s$
- If we take a canoe trip, then we will be home by sunset. $s \rightarrow t$

Denote:

- p = It is sunny this afternoon
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Contradiction and Tautology

Some composite sentences may always (under any interpretation) evaluate to a single truth value:

- **Contradiction** (always ***False***)

$$P \wedge \neg P$$

- **Tautology** (always ***True***)

$$P \vee \neg P$$

$$\left.\begin{array}{l} \neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q) \\ \neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q) \end{array}\right\} \text{DeMorgan's Laws}$$

Model, validity and satisfiability

- A **model (in logic):** An **interpretation is a model** for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  – There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  – i.e., if its negation is **not satisfiable** (leads to contradiction)

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

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True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

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Satisfiable sentence

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

Model, validity and satisfiability

- A **model (in logic):** An interpretation is a model for a set of sentences if it assigns true to each sentence in the set.
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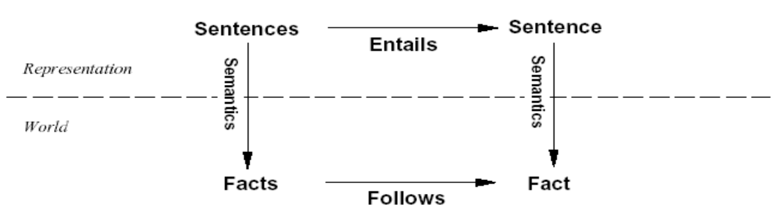
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

			Satisfiable sentence	**Valid sentence**
P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	***True***	***True***	***False***	***True***
True	***False***	***True***	***True***	***True***
False	***True***	***True***	***False***	***True***
False	***False***	***False***	***False***	***True***

Entailment

- **Entailment** reflects the relation of one fact in the world following from the others according to logic



- Entailment $KB \models \alpha$
- Knowledge base KB entails sentence α if and only if α is true in all worlds where KB is true

Sound and complete inference

Inference is a process by which conclusions are reached.

- We want to implement the inference process on a computer !!

Assume an ***inference procedure i*** that

- derives a sentence α from the KB : $KB \vdash_i \alpha$

Properties of the inference procedure in terms of entailment

- **Soundness:** An inference procedure is **sound**

  If $KB \vdash_i \alpha$ then it is true that $KB \models \alpha$

- **Completeness:** An inference procedure is **complete**

  If $KB \models \alpha$ then it is true that $KB \vdash_i \alpha$

Logical inference problem

Logical inference problem:

- **Given:**
  - a knowledge base KB (a set of sentences) and
  - a sentence α (called **a theorem)**,
- **Does a KB semantically entail** α ? $KB \models \alpha$?

In other words: In all interpretations in which sentences in the KB are true, is also α true?

Question: Is there a procedure (program) that can decide this problem in a finite number of steps?

Answer: **Yes**. Logical inference problem for the propositional logic is **decidable**.

Solving logical inference problem

In the following:

How to design the procedure that answers:

$$KB \models \alpha \ ?$$

Three approaches:

- **Truth-table approach**
- **Inference rules**
- **Conversion to the inverse SAT problem**
  - **Resolution-refutation**

Truth-table approach

Problem: $KB \models \alpha \ ?$

- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False values to propositional symbols)

Example:

		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

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		KB		α
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$
True	*True*	*True*	*True*	*True*
True	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*False*
False	*False*	*False*	*True*	*False*

Truth-table approach

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- We need to check all possible interpretations for which the KB is true (models of KB) whether α is true for each of them

Truth table:

- enumerates truth values of sentences for all possible interpretations (assignments of True/False to propositional symbols)

Example:

		KB		α	
P	Q	$P \vee Q$	$P \Leftrightarrow Q$	$(P \vee \neg Q) \wedge Q$	
True	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*False*	*False*	
False	*True*	*True*	*False*	*False*	
False	*False*	*False*	*True*	*False*	

Truth-table approach

A two steps procedure:

1. **Generate table for all possible interpretations**
2. Check whether the sentence α evaluates to true whenever KB evaluates to true

Example: $KB = (A \lor C) \land (B \lor \neg C) \qquad \alpha = (A \lor B)$

A	B	C	$A \lor C$	$(B \lor \neg C)$	KB	α
True	*True*	*True*				
True	*True*	*False*				
True	*False*	*True*				
True	*False*	*False*				
False	*True*	*True*				
False	*True*	*False*				
False	*False*	*True*				
False	*False*	*False*				

Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. **Check whether the sentence α evaluates to true whenever *KB* evaluates to true**

Example: $KB = (A \lor C) \land (B \lor \neg C) \qquad \alpha = (A \lor B)$

A	B	C	$A \lor C$	$(B \lor \neg C)$	KB	α
True	*True*	*True*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

Truth-table approach

A two steps procedure:

1. Generate table for all possible interpretations
2. **Check whether the sentence α evaluates to true whenever *KB* evaluates to true**

Example: $KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α	
True	*True*	*True*	*True*	*True*	*True*	*True*	✓
True	*True*	*False*	*True*	*True*	*True*	*True*	✓
True	*False*	*True*	*True*	*False*	*False*	*True*	
True	*False*	*False*	*True*	*True*	*True*	*True*	✓
False	*True*	*True*	*True*	*True*	*True*	*True*	✓
False	*True*	*False*	*False*	*True*	*False*	*True*	
False	*False*	*True*	*True*	*False*	*False*	*False*	
False	*False*	*False*	*False*	*True*	*False*	*False*	

Truth-table approach

$KB = (A \vee C) \wedge (B \vee \neg C)$ $\alpha = (A \vee B)$

A	B	C	$A \vee C$	$(B \vee \neg C)$	KB	α
True	*True*	*True*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*True*	*True*	*False*	*False*	*True*
True	*False*	*False*	*True*	*True*	*True*	*True*
False	*True*	*True*	*True*	*True*	*True*	*True*
False	*True*	*False*	*False*	*True*	*False*	*True*
False	*False*	*True*	*True*	*False*	*False*	*False*
False	*False*	*False*	*False*	*True*	*False*	*False*

KB entails α

- The **truth-table approach** is **sound and complete** for the propositional logic!!

Limitations of the truth table approach.

$$KB \models \alpha \ ?$$

What is the computational complexity of the truth table approach?

Exponential in the number of the proposition symbols

2^n Rows in the table must be filled

Observation: typically only a small subset of rows is true

Question: Is it possible to make the process more efficient?

Solution: Inference rules

- checks only entries for which KB is *True.*
- Represent sound inference patterns repeated in inferences
- Can be used to generate new (sound) sentences from the existing ones

Inference rules for logic

- **Modus ponens**

$$\frac{A \Rightarrow B, \quad A}{B}$$

$\leftarrow$ **premise**

$\leftarrow$ **conclusion**

- If both sentences in the premise are true then conclusion is true.
- The modus ponens inference rule is **sound.**
  - We can prove this through the truth table.

A	B	$A \Rightarrow B$
False	*False*	*True*
False	*True*	*True*
True	*False*	*False*
True	*True*	*True*

Inference rules for logic

- **And-elimination**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

- **And-introduction**

$$\frac{A_1, A_2, \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$

- **Or-introduction**

$$\frac{A_i}{A_1 \vee A_2 \vee \ldots A_i \vee \quad A_n}$$

Inference rules for logic

- **Elimination of double negation**

$$\frac{\neg\neg A}{A}$$

- **Unit resolution**

$$\frac{A \vee B, \quad \neg A}{B}$$

A special case of

- **Resolution**

$$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

- All of the above inference rules **are sound.** We can prove this through the truth table, similarly to the **modus ponens** case.

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P **From 1 and And-elim**

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R

From 2,4 and Modus ponens

$$\frac{A \Rightarrow B, \quad A}{B}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q

From 1 and And-elim

$$\frac{A_1 \wedge A_2 \wedge \quad A_n}{A_i}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$

From 5,6 and And-introduction

$$\frac{A_1, A_2, \quad A_n}{A_1 \wedge A_2 \wedge \quad A_n}$$

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P
5. R
6. Q
7. $(Q \wedge R)$
8. S

$$\frac{A \Rightarrow B, \quad A}{B}$$

From 7,3 and Modus ponens

Proved: S

Example. Inference rules approach.

KB: $P \wedge Q \quad P \Rightarrow R \quad (Q \wedge R) \Rightarrow S$ **Theorem:** S

1. $P \wedge Q$
2. $P \Rightarrow R$
3. $(Q \wedge R) \Rightarrow S$
4. P **From 1 and And-elim**
5. R **From 2,4 and Modus ponens**
6. Q **From 1 and And-elim**
7. $(Q \wedge R)$ **From 5,6 and And-introduction**
8. S **From 7,3 and Modus ponens**

Proved: S