CS 2740 Knowledge representation
Lecture 20

Bayesian belief networks: Inference

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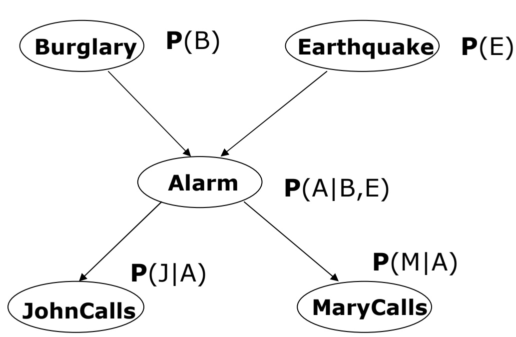
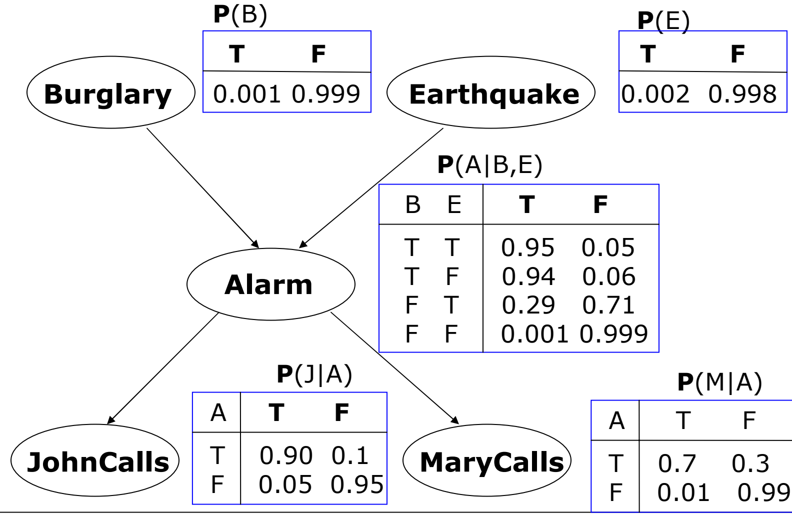
Bayesian belief network.

1. **Directed acyclic graph**
   - **Nodes** = random variables
   - **Links** = missing links encode independences.



Bayesian belief network

2. **Local conditional distributions**
   - relate variables and their parents



P(B)

T	F
0.001	0.999

P(E)

T	F
0.002	0.998

P(A|B,E)

B	E	T	F
T	T	0.95	0.05
T	F	0.94	0.06
F	T	0.29	0.71
F	F	0.001	0.999

P(J|A)

A	T	F
T	0.90	0.1
F	0.05	0.95

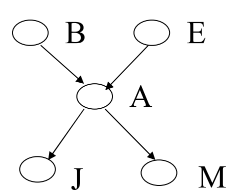
P(M|A)

A	T	F
T	0.7	0.3
F	0.01	0.99

Full joint distribution in BBNs

Full joint distribution is defined in terms of local conditional distributions (obtained via the chain rule):

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$



Example:

Assume the following assignment of values to random variables

$B = T, E = T, A = T, J = T, M = F$

Then its probability is:

$$P(B=T, E=T, A=T, J=T, M=F) =$$
$$P(B=T)P(E=T)P(A=T \mid B=T, E=T)P(J=T \mid A=T)P(M=F \mid A=T)$$

Parameter complexity problem

- In the BBN the **full joint distribution** is defined as:

$$\mathbf{P}(X_1, X_2, \ldots, X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

Alarm example: 5 binary (True, False) variables

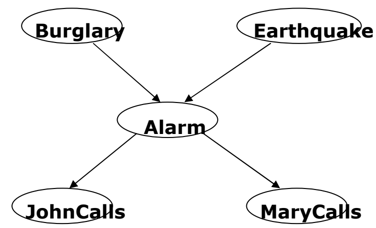
# of parameters of the full joint:

$$2^5 = 32$$

One parameter is for free:

$$2^5 - 1 = 31$$

# of parameters of the BBN: ?



Parameter complexity problem

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$$\mathbf{P}(X_1, X_2, \ldots, X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

Alarm example: 5 binary (True, False) variables

# of parameters of the full joint:

$$2^5 = 32$$

One parameter is for free:

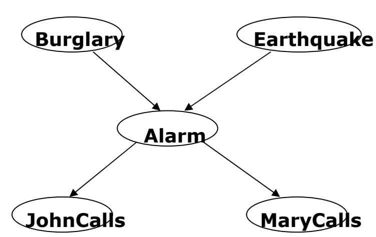
$$2^5 - 1 = 31$$

# of parameters of the BBN:

$$2^3 + 2(2^2) + 2(2) = 20$$

One parameter in every conditional is for free:

?



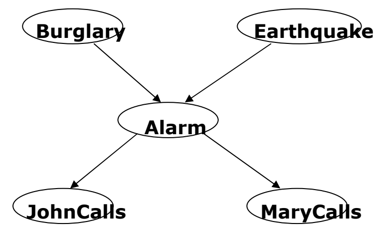
Parameter complexity problem

- In the BBN the **full joint distribution** is defined as:

$$\mathbf{P}(X_1, X_2, .., X_n) = \prod_{i=1,..n} \mathbf{P}(X_i \mid pa(X_i))$$

- **What did we save?**

Alarm example: 5 binary (True, False) variables



# of parameters of the full joint:

$$2^5 = 32$$

One parameter is for free:

$$2^5 - 1 = 31$$

# of parameters of the BBN:

$$2^3 + 2(2^2) + 2(2) = 20$$

One parameter in every conditional is for free:

$$2^2 + 2(2) + 2(1) = 10$$

Inference in Bayesian networks

- BBN models compactly the full joint distribution by taking advantage of existing independences between variables
  - Smaller number of parameters
- But we are interested in solving various **inference tasks**:
  - **Diagnostic task. (from effect to cause)**

    $$\mathbf{P}(Burglary \mid JohnCalls = T)$$

  - **Prediction task. (from cause to effect)**

    $$\mathbf{P}(JohnCalls \mid Burglary = T)$$

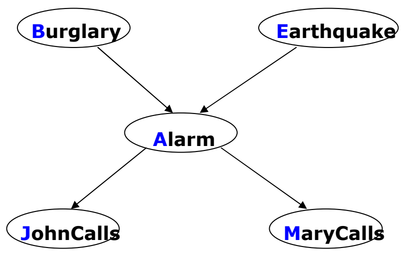
  - **Other probabilistic queries** (queries on joint distributions).

    $$\mathbf{P}(Alarm)$$

- **Question:** Can we take advantage of independences to construct special algorithms and speedup the inference?

Inference in Bayesian network

- **Bad news:**
  - Exact inference problem in BBNs is NP-hard (Cooper)
  - Approximate inference is NP-hard (Dagum, Luby)
- **But** very often we can achieve significant improvements
- Assume our Alarm network



- Assume we want to compute: $P(J = T)$

Inference in Bayesian networks

Computing: $P(J = T)$

Approach 1. Blind approach.

- Sum out all un-instantiated variables from the full joint,
- express the joint distribution as a product of conditionals

$$P(J = T) =$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(B = b, E = e, A = a, J = T, M = m)$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J = T \mid A = a) P(M = m \mid A = a) P(A = a \mid B = b, E = e) P(B = b) P(E = e)$$

Computational cost:

Number of additions: ?

Number of products: ?

Inference in Bayesian networks

Computing: $P(J = T)$

Approach 1. Blind approach.

- Sum out all un-instantiated variables from the full joint,
- express the joint distribution as a product of conditionals

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$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(B = b, E = e, A = a, J = T, M = m)$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J = T \mid A = a) P(M = m \mid A = a) P(A = a \mid B = b, E = e) P(B = b) P(E = e)$$

Computational cost:

Number of additions: 15

Number of products: ?

Inference in Bayesian networks

Computing: $P(J = T)$

Approach 1. Blind approach.

- Sum out all un-instantiated variables from the full joint,
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$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(B = b, E = e, A = a, J = T, M = m)$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J = T \mid A = a) P(M = m \mid A = a) P(A = a \mid B = b, E = e) P(B = b) P(E = e)$$

Computational cost:

Number of additions: 15

Number of products: 16*4=64

Inference in Bayesian networks

Approach 2. Interleave sums and products

- Combines sums and product in a smart way (multiplications by constants can be taken out of the sum)

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

$$=\sum_{b\in T,F}\sum_{a\in T.F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(B=b)[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)]$$

$$=\sum_{a\in T,F}P(J=T\mid A=a)[\sum_{m\in T,F}P(M=m\mid A=a)][\sum_{b\in T,F}P(B=b)[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)]]$$

Computational cost:

Number of additions: 1+2*[1+1+2*1]=**?**

Number of products: 2*[2+2*(1+2*1)]=?

Inference in Bayesian networks

Approach 2. Interleave sums and products

- Combines sums and product in a smart way (multiplications by constants can be taken out of the sum)

$$P(J=T)=$$

$$=\sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

$$=\sum_{b\in T,F}\sum_{a\in T.F}\sum_{m\in T,F}P(J=T\mid A=a)P(M=m\mid A=a)P(B=b)[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)]$$

$$=\sum_{a\in T,F}P(J=T\mid A=a)[\sum_{m\in T,F}P(M=m\mid A=a)][\sum_{b\in T,F}P(B=b)[\sum_{e\in T,F}P(A=a\mid B=b,E=e)P(E=e)]]$$

Computational cost:

Number of additions: 1+2*[1+1+2*1]=**9**

Number of products: 2*[2+2*(1+2*1)]=?

Inference in Bayesian networks

Approach 2. Interleave sums and products

- Combines sums and product in a smart way (multiplications by constants can be taken out of the sum)

$P(J=T) =$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J=T \mid A=a) P(M=m \mid A=a) P(A=a \mid B=b, E=e) P(B=b) P(E=e)$$

$$= \sum_{b \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J=T \mid A=a) P(M=m \mid A=a) P(B=b) [\sum_{e \in T,F} P(A=a \mid B=b, E=e) P(E=e)]$$

$$= \sum_{a \in T,F} P(J=T \mid A=a) [\sum_{m \in T,F} P(M=m \mid A=a)] [\sum_{b \in T,F} P(B=b) [\sum_{e \in T,F} P(A=a \mid B=b, E=e) P(E=e)]]$$

Computational cost:

Number of additions: 1+2*[1+1+2*1]=**9**

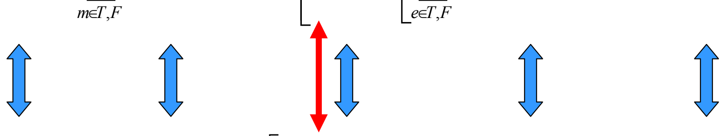
Number of products: 2*[2+2*(1+2*1)]=16

Inference in Bayesian networks

- The smart interleaving of sums and products can help us to speed up the computation of joint probability queries
- What if we want to compute: $P(B=T, J=T)$

$P(B=T, J=T) =$

$$= \sum_{a \in T,F} P(J=T \mid A=a) [\sum_{m \in T,F} P(M=m \mid A=a)] \left[P(B=T) \left[\sum_{e \in T,F} P(A=a \mid B=T, E=e) P(E=e) \right] \right]$$



$P(J=T) =$

$$= \sum_{a \in T,F} P(J=T \mid A=a) [\sum_{m \in T,F} P(M=m \mid A=a)] \left[\sum_{b \in T,F} P(B=b) \left[\sum_{e \in T,F} P(A=a \mid B=b, E=e) P(E=e) \right] \right]$$

- A lot of shared computation
  – Smart cashing of results can save the time for more queries

Inference in Bayesian network

- **Exact inference algorithms:**
  - **Variable elimination**
  - Recursive decomposition (Cooper, Darwiche)
  - Belief propagation algorithm (Pearl)
  - Arc reversal (Olmsted, Schachter)
- **Approximate inference algorithms:**
  - **Monte Carlo methods:**
    - Forward sampling, Likelihood sampling
  - Variational methods

Variable elimination

- **Variable elimination:**
  - Interleave sum and products one variable at the time during the inference
  - Typically relies on a special structure (called **joint tree**) that groups together multiple variables
  - E.g. Query $P(J=T)$ requires to eliminate A,B,E,M and this can be done in different order

$$P(J=T) =$$

$$= \sum_{b \in T,F} \sum_{e \in T,F} \sum_{a \in T,F} \sum_{m \in T,F} P(J=T \mid A=a) P(M=m \mid A=a) P(A=a \mid B=b, E=e) P(B=b) P(E=e)$$

Variable elimination

Assume the elimination order: M, E, B,A to calculate $P(J=T)$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F}\sum_{m\in T,F} P(J=T\mid A=a)P(M=m\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e)\left[\sum_{m\in T,F} P(M=m\mid A=a)\right]$$

$$= \sum_{b\in T,F}\sum_{e\in T,F}\sum_{a\in T,F} P(J=T\mid A=a)P(A=a\mid B=b,E=e)P(B=b)P(E=e) \quad 1$$

$$= \sum_{a\in T,F}\sum_{b\in T,F} P(J=T\mid A=a)P(B=b)\left[\sum_{e\in T,F} P(A=a\mid B=b,E=e)P(E=e)\right]$$

$$= \sum_{a\in T,F}\sum_{b\in T,F} P(J=T\mid A=a)P(B=b)\,\tau_1(A=a,B=b)$$

$$= \sum_{a\in T,F} P(J=T\mid A=a)\left[\sum_{e\in T,F} P(B=b)\,\tau_1(A=a,B=b)\right]$$

$$= \sum_{a\in T,F} P(J=T\mid A=a) \quad \tau_2(A=a)$$

Factors

- **Factor:** is a function that maps value assignments for a subset of random variables to $\Re$ (reals)
- **The scope of the factor:**
  - a set of variables defining the factor
- **Example:**
  - Assume discrete random variables x (with values a1,a2, a3) and y (with values b1 and b2)
  - Factor:

$$\phi(x, y)$$

a1	b1	0.5
a1	b2	0.2
a2	b1	0.1
a2	b2	0.3
a3	b1	0.2
a3	b2	0.4

  - Scope of the factor:

$$\{x, y\}$$

Factor Product

$$\phi_1(x,y)\phi_2(y,z) = \tau(x,y,z)$$

a1	b1	0.5
a1	b2	0.2
a2	b1	0.1
a2	b2	0.3
a3	b1	0.2
a3	b2	0.4

•

b1	c1	0.1
b1	c2	0.6
b2	c1	0.3
b2	c2	0.4

=

a1	b1	c1	0.5*0.1
a1	b1	c2	0.5*0.6
a1	b2	c1	0.2*0.3
a1	b2	c2	0.2*0.4
a2	b1	c1	0.1*0.1
a2	b1	c2	0.1*0.6
a2	b2	c1	0.3*0.3
a2	b2	c2	0.3*0.4
a3	b1	c1	0.2*0.1
a3	b1	c2	0.2*0.6
a3	b2	c1	0.4*0.3
a3	b2	c2	0.4*0.4

Factor Product

$$\phi_1(x,y)\phi_2(y,z) = \tau(x,y,z)$$

a1	b1	0.5
a1	b2	0.2
a2	b1	0.1
a2	b2	0.3
a3	b1	0.2
a3	b2	0.4

•

b1	c1	0.1
b1	c2	0.6
b2	c1	0.3
b2	c2	0.4

=

a1	b1	c1	0.5*0.1
a1	b1	c2	0.5*0.6
a1	b2	c1	0.2*0.3
a1	b2	c2	0.2*0.4
a2	b1	c1	0.1*0.1
a2	b1	c2	0.1*0.6
a2	b2	c1	0.3*0.3
a2	b2	c2	0.3*0.4
a3	b1	c1	0.2*0.1
a3	b1	c2	0.2*0.6
a3	b2	c1	0.4*0.3
a3	b2	c2	0.4*0.4

Factor Sum (marginalization)

$$\sum_y \phi(x, y, z) = \tau(x, z)$$

$\sum$

a1	b1	c1	0.2
a1	b1	c2	0.35
a1	b2	c1	0.4
a1	b2	c2	0.15
a2	b1	c1	0.5
a2	b1	c2	0.1
a2	b2	c1	0.3
a2	b2	c2	0.2
a3	b1	c1	0.25
a3	b1	c2	0.45
a3	b2	c1	0.15
a3	b2	c2	0.25

=

a1	c1	0.6
a1	c2	0.5
a2	c1	0.8
a2	c2	0.3
a3	c1	0.4
a3	c2	0.7

Factor Sum (marginalization)

$$\sum_y \phi(x, y, z) = \tau(x, z)$$

$\sum$

a1	b1	c1	0.2
a1	b1	c2	0.35
a1	b2	c1	0.4
a1	b2	c2	0.15
a2	b1	c1	0.5
a2	b1	c2	0.1
a2	b2	c1	0.3
a2	b2	c2	0.2
a3	b1	c1	0.25
a3	b1	c2	0.45
a3	b2	c1	0.15
a3	b2	c2	0.25

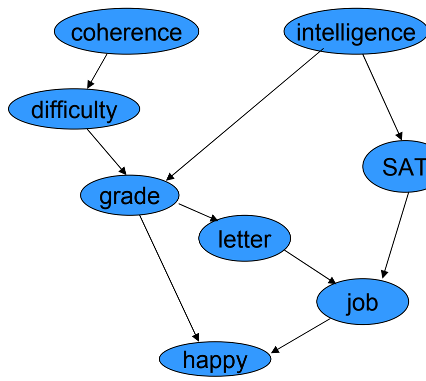
=

a1	c1	0.6
a1	c2	0.5
a2	c1	0.8
a2	c2	0.3
a3	c1	0.4
a3	c2	0.7

Variable elimination

The order in which variables are eliminated may effect the efficiency of the variable elimination process

Assume the following BBN and calculation of *P(Job)*:



Variable elimination

Calculations performed in terms of factors:

$$p(J) = \sum_{L,S,G,H,I,D,C} \phi(c)\phi(i)\phi(d,c)\phi(g,i,d)\phi(s,i)\phi(l,g)\phi(j,l,s)\phi(h,g,j)$$

$$= \sum_{L,S,G,H,I,D} \phi(i)\phi(g,i,d)\phi(s,i)\phi(l,g)\phi(j,l,s)\phi(h,g,j)\sum_{C}\phi(c)\phi(d,c)$$

$$= \sum_{L,S,G,H,I,D} \phi(i)\phi(g,i,d)\phi(s,i)\phi(l,g)\phi(j,l,s)\phi(h,g,j)\tau(d)$$

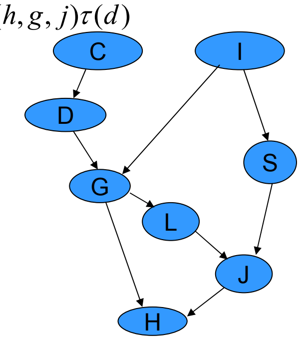
$$\dots$$

$$= \sum_{L,S}\phi(j,l,s)\sum_{G}\phi(l,g)\tau(s,g)\tau(g,j)$$

$$= \sum_{L,S}\phi(j,l,s)\tau(l,s,j)$$

$$= \sum_{L}\tau(l,j)$$

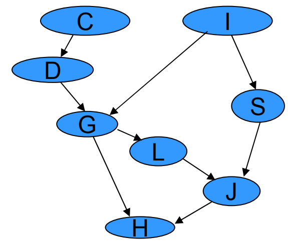
$$= \tau(j)$$



Variable elimination

Trace 1:

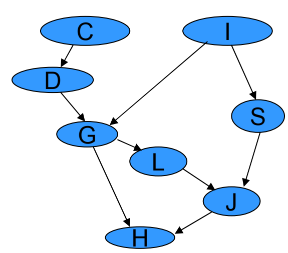
Step	Var	Factors Used	New Factor
1	C	$\phi_c(C), \phi_D(D, C)$	$\tau_1(D)$
2	D	$\phi_G(G, I, D), \tau_1(D)$	$\tau_2(G, I)$
3	I	$\phi_I(I), \phi_S(S, I), \tau_2(G, I)$	$\tau_3(G, S)$
4	H	$\phi_H(H, G, J)$	$\tau_4(G, J)$
5	G	$\tau_4(G, J), \tau_3(G, S), \phi_L(L, G)$	$\tau_5(J, L, S)$
6	S	$\tau_5(J, L, S), \phi_J(J, L, S)$	$\tau_6(J, L)$
7	L	$\tau_6(J, L)$	$\tau_7(J)$



Variable elimination

Trace 1:

Step	Var	Factors Used	New Factor
1	C	$\phi_c(C), \phi_D(D, C)$	$\tau_1(D)$
2	D	$\phi_G(G, I, D), \tau_1(D)$	$\tau_2(G, I)$
3	I	$\phi_I(I), \phi_S(S, I), \tau_2(G, I)$	$\tau_3(G, S)$
4	H	$\phi_H(H, G, J)$	$\tau_4(G, J)$
5	G	$\tau_4(G, J), \tau_3(G, S), \phi_L(L, G)$	$\tau_5(J, L, S)$
6	S	$\tau_5(J, L, S), \phi_J(J, L, S)$	$\tau_6(J, L)$
7	L	$\tau_6(J, L)$	$\tau_7(J)$

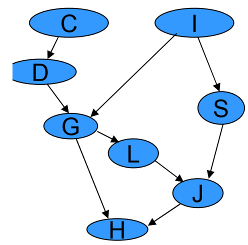


Complexity: 4 variables – 1 summed away

Variable elimination

Trace 2:

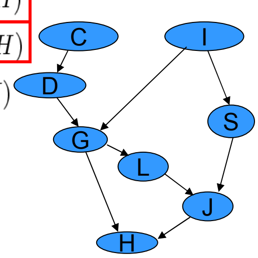
Step	Var	Factors Used	New Factor
1	G	$\phi_G(G,I,D), \phi_L(L,G)\phi_H(H,G,J)$	$\tau_1(I,D,L,J,H)$
2	I	$\phi_I(I), \phi_S(S,I)\tau_1(I,D,L,J,H)$	$\tau_2(D,L,S,J,H)$
3	S	$\phi_J(J,L,S), \tau_2(D,L,S,J,H)$	$\tau_3(D,L,J,H)$
4	L	$\tau_3(D,L,J,H)$	$\tau_4(D,J,H)$
5	H	$\tau_4(D,J,H)$	$\tau_5(D,J)$
6	C	$\tau_5(D,J), \phi_D(D,C)$	$\tau_6(D,J)$
7	D	$\tau_6(D,J)$	$\tau_7(J)$



Variable elimination

Trace 2:

Step	Var	Factors Used	New Factor
1	G	$\phi_G(G,I,D), \phi_L(L,G)\phi_H(H,G,J)$	$\tau_1(I,D,L,J,H)$
2	I	$\phi_I(I), \phi_S(S,I)\tau_1(I,D,L,J,H)$	$\tau_2(D,L,S,J,H)$
3	S	$\phi_J(J,L,S), \tau_2(D,L,S,J,H)$	$\tau_3(D,L,J,H)$
4	L	$\tau_3(D,L,J,H)$	$\tau_4(D,J,H)$
5	H	$\tau_4(D,J,H)$	$\tau_5(D,J)$
6	C	$\tau_5(D,J), \phi_D(D,C)$	$\tau_6(D,J)$
7	D	$\tau_6(D,J)$	$\tau_7(J)$



Complexity: 6 variables used – 1 summed out