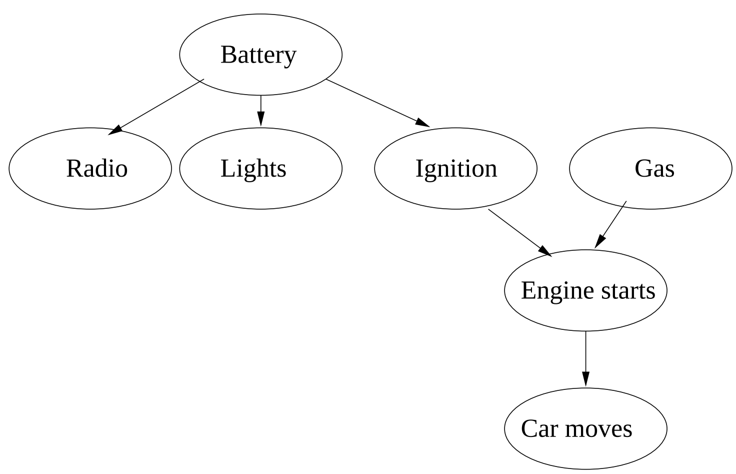
University of Pittsburgh
CS 2740 Knowledge Representation — Handout 16
Professor Milos Hauskrecht — December 5, 2008

Problem set 7 solutions

Uncertainty.

Problem 1. Bayesian Belief Networks.

Assume the Bayesian belief network for the diagnosis of car's electircal system.



Assume that all variables in the network are binary with True and False values. To simplify the formulas we use first letters of ranom variables below, that is, R for Radio, C for Car moves, etc.

Part a. The belief network structure encodes conditional independence relations in graphical terms. Give at least four examples of conditional and one example of marginal indepen- dences encoded in the network structure.

Examples of conditional independences:

- Engine starts is independent of Battery given Ignition;

- Engine starts is independent of Lights given Battery;
- Radio is independent of Gas given Ignition;
- Lights is independent of Car Moves given Battery.

An example of the marginal independence:
Battery and Gas are (marginally) independent.

Complexity

Part a. Assume that all variables in the network are binary (have two possible values). What is the number of probabilities of the full joint distribution?

Answer: If the belief network has n variables, and each variable is binary, then the full joint is defined with 2^n parameters. Since we have a total of 7 variables in our problem the full joint consists of 128 parameters.

Note that we can always get one parameter of of the probability distribution for free, since we know that all probabilities must sum to 1.

Part b. Again assume that all the variables are binary. What is the number of parameters defining the belief network in the figure?

Paramteres of the belief network are conditional probabilities relating variables and their parents. Therefore the total number of parameters needed is:

$$\begin{aligned}
= \quad & 2(\text{for Battery}) + 2 \times 2(\text{for Radio given Battery}) + 2 \times 2(\text{for Light given Battery}) \\
& +2 \times 2(\text{for Ignition given Battery}) + 2(\text{for Gas}) + 2 \times 4(\text{for Engine given Ignition and Gas}) \\
& +2 \times 2(\text{for Moves given Engine}) \\
= \quad & 28
\end{aligned}$$

Note that if we use the fact that probabilities sum to 1, the minimum number of parameters needed to define the belief network is:

$$\begin{aligned}
= \quad & 1(\text{for Battery}) + 2 \times 1(\text{for Radio given Battery}) + 2 \times 1(\text{for Light given Battery}) \\
& +2 \times 1(\text{for Ignition given Battery}) + 2(\text{for Gas}) + 2 \times 2(\text{for Engine given Ignition and Gas}) \\
& +2 \times 1(\text{for Moves given Engine}) \\
= \quad & 14
\end{aligned}$$

Part c. Show how to compute the full joint probability over variables using the Bayesian belief network and its parameters. Assume we are interested in calculating the joint probability for: Battery=T, Radio=F,Light=T,Ignition=T, Gas=T, EngineStarts=F,Carmoves=F.

Answer: The full joint probability is joint distribution over all the values in the domain. In the Bayesian belief networks it is computed as a product of conditional probabilities for every node-parent combination. That is:

$$P(X_1, X_2, \ldots X_n) = \prod_{i=1}^{n} P(X_i | Parents(X_i))$$

The joint probability $P(Battery = T, Radio = T, Lights = T, Ignition = T, Gas = F, Engine = T, Moves = F)$ is then computed as:

$$\begin{aligned} = \quad & P(Battery = T) \times P(Radio = T | Battery = T) \times P(Light = T | Battery = T) \\ & \times P(Ignition = T | Battery = T) \times P(Gas = F) \times P(Engine = T | Gas = F, Ignition = T) \\ & \times P(Moves = F | Engine = T) \end{aligned}$$

Inferences

Part a. Assume we want to compute the probability of Car not moving, that is $P(Carmoves = False)$. Write down the expression for computing the probability from conditionals via blind approach. What is the inference cost? The inference cost should be expressed in terms of the number of additions and the number of products used.

Answer: To simplify the formulas we use only first letters of random variables, that is, R for Radio, C for Car moves, etc. The target probability $P(C = F)$ can be written by marginalizing (summing out) all variables not in the expression from the full joint probability. That is

$$\begin{aligned} & P(C = F) = \\ & = \sum_{b \in \{T,F\}} \sum_{r \in \{T,F\}} \sum_{l \in \{T,F\}} \sum_{i \in \{T,F\}} \sum_{g \in \{T,F\}} \sum_{e \in \{T,F\}} P(B = b) \times P(R = r | B = b) \times \\ & \quad \times P(L = l | B = b) \times P(I = i | B = b) \times P(G = g) \times P(E = e | G = g, I = i) \times P(C = T | E = e) \end{aligned}$$

The inference cost for the blind approach comes up to:
Number of additions: $2^6 - 1 = 63$
Number of products: $64 \times 6 = 384$

Part b. Propose a more efficient solution for computing probability $P(C = F)$ that interleaves sums and products. Write down the new expression and give its inference cost.

Answer: We can speed up the computation of the probability $P(C = F)$ by interleaving sums and products of the above expression. One way to interleave the expressions is as follows:

$$\begin{aligned} & P(C = F) = \\ & = \sum_{b \in \{T,F\}} P(B = b) \times \end{aligned}$$

$$\times \left[\sum_{r \in \{T,F\}} P(R = r | B = b) \right]$$

$$\times \left[\sum_{l \in \{T,F\}} P(L = l | B = b) \right]$$

$$\times \left[\sum_{i \in \{T,F\}} P(I = i | B = b) \times \left\{ \sum_{g \in \{T,F\}} P(G = g) \times \left(\sum_{e \in \{T,F\}} P(E = e | G = g, I = i) \times P(C = F | E = e) \right) \right\} \right]$$

The inference cost for the new expression is:
Number of additions: $1 + 2 * (1 + 1 + 1) + 2 * 2 * (1) + 2 * 2 * 2 * (1) = 1 + 6 + 4 + 8 = 19$
Number of products: $2 * (3) + 2 * 2 * (1) + 2 * 2 * 2 * (1) + 2 * 2 * 2 * 2 * (1) = 6 + 4 + 8 + 16 = 34$.

Note that we can further speed up the computation of $P(C = False)$ by further noticing the we can directly eliminate the sums over Radio and Lights values since they must add up to 1, that is:

$$\left[\sum_{r \in \{T,F\}} P(R = r | B = b) \right] = 1,$$

$$\left[\sum_{l \in \{T,F\}} P(L = l | B = b) \right] = 1.$$

After the emimination we get:

$P(C = F) =$

$$= \left[\sum_{b \in \{T,F\}} P(B = b) \times \left[\sum_{i \in \{T,F\}} P(I = i | B = b) \times \left[\sum_{g \in \{T,F\}} P(G = g) \times \left[\sum_{e \in \{T,F\}} P(E = e | G = g, I = i) \times P(C = F | E = e) \right] \right] \right] \right]$$

Inference cost for the improved expression is then:
Number of additions: $1 + 2 * (1) + 2 * 2 * (1) + 2 * 2 * 2 * (1) = 13$
Number of products: $2 * 1 + 2 * 2 * (1) + 2 * 2 * 2 * (1) + 2 * 2 * 2 * 2 * (1) = 30$

Part c.