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The Gibbard–Satterthwaite theorem: a simple proof

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Abstract

A simple proof of the Gibbard–Satterthwaite theorem is provided.

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The classic Gibbard–Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) states (essentially) that a dictatorship is the only non-manipulable voting mechanism. This theorem is intimately connected to Arrow's impossibility theorem. In this note, I provide a simple proof of the theorem, adapting an idea used by Geanakoplos (1996) in one of his simple proofs of Arrow's theorem.[1]

Consider a set of individuals N, with $|N| = n$, and a set of alternatives M, with $|M| = m$. We assume that each person has a (strict) linear preference ordering over the alternatives. A **social choice function** selects a single alternative on the basis of the *reported preference profile* of the individuals. The rule is **unanimous** if it always selects an alternative that is top-ranked by everyone. It is **strategyproof** if it is a dominant strategy for an individual with any permissible ranking to truthfully report her preferences.

Theorem. (Gibbard–Satterthwaite) *Suppose there are at least three alternatives and that for each*

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[1] Philip Reny (1999) further emphasizes the connection between the Gibbard–Satterthwaite theorem and Arrow's theorem by ingeniously providing a 'shared' simple proof for both results. His paper was written independently of this one, but was also inspired by Geanakoplos (1996). Sen (2000) provides a direct proof that does not require the universal domain of preferences. See also Barberà (1983) for an earlier relatively simple proof.

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individual any strict ranking of these alternatives is permissible. Then the only unanimous, strategy-proof social choice function is a dictatorship.

Proof. Suppose we are given a unanimous, strategy-proof social choice function. We show that one of the voters must be a dictator.

We first observe the following fact:

- **Fact 1:** Suppose that alternative A is selected given some preference profile. Modify the profile by raising some alternative X in individual i's ranking (holding everything else fixed). Then either A or X is now selected.

Suppose instead that when X rises, some alternative C different from A and X is chosen. Then, if i prefers A to C he would not report the change, whereas if he prefers C to A he would have falsely reported this change earlier.

Step 1. We begin with an arbitrary strict profile in which everyone ranks B last. Alternative B is not selected, or else, by strategyproofness, it would still have to be selected if the voters, one at a time, raised A to the top of their profiles, contradicting unanimity. Now, starting with individual 1 and continuing in order with the other voters one at a time, have B jump from the bottom of each ranking to the top (leaving the other relative rankings in place). Let r be the *pivotal* individual for whom the jump causes B to be selected. That is, with the profile:

$$\begin{array}{cccccccc}
\mathbf{1} & \mathbf{2} & \cdots & \mathbf{r-1} & \mathbf{r} & \mathbf{r+1} & \cdots & \mathbf{n} \\
B & B & B\cdots & B & K & A & \cdots & C \\
D & E & & \vdots & \vdots & & & \vdots \\
 & & & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & & & & \\
T & F & & S & B & B & B\cdots & B
\end{array} \qquad \text{(Profile 1)}$$

B is not selected, whereas with the profile:

$$\begin{array}{cccccccc}
\mathbf{1} & \mathbf{2} & \cdots & \mathbf{r-1} & \mathbf{r} & \mathbf{r+1} & \cdots & \mathbf{n} \\
B & B & B\cdots & B & B & A & \cdots & C \\
D & E & & \vdots & K & & & \vdots \\
 & & & & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & & & & \\
T & F & & S & & B & B\cdots & B
\end{array} \qquad \text{(Profile 2)}$$

B is selected.

Consider Profile 2, where B is selected. Alternative B must still be selected if any player $i \geq r+1$ submits a different ranking, or else i would misrepresent. Also, B must still be selected if any player $i \leq r$ submits a different ranking with B still ranked first or else i would not honestly report this ranking. Considering the players one at a time we have:

- B is chosen whenever the first r players rank B first. (* B)

Consider Profile 1 where B is not chosen. Alternative B must still not be chosen if any player

$i \leq r - 1$ submits a different ranking, or else i would do just that. Also B must still not be chosen if any player $i \geq r$ submits a different ranking with B ranked last, or else i would not honestly report this ranking. Considering the players one at a time we have:

- B is not selected whenever voters r through n rank alternative B last. (**Not B)

We will show that the pivotal individual r is, in fact, a dictator.

Step 2. Consider any profile of the form

1	**2**	· · ·	**r − 1**	**r**	**r + 1**	· · ·	**n**
?	?	? · · ·	?	K	?	· · ·	?
?	?		⋮				⋮
				⋮	⋮		⋮
⋮	⋮		⋮				
B	B	B	B	B	B	B · · ·	B

(Profile 3)

First, raise K to the top position for all voters. By unanimity K is chosen. Now raise B to the top for voters 1 through $r - 1$ one at a time, leaving B on the bottom for the remaining voters:

1	**2**	· · ·	**r − 1**	**r**	**r + 1**	· · ·	**n**
B	B	B · · ·	B	K	K	K · · ·	K
K	K	K · · ·	K	?	?	? · · ·	?
				⋮	⋮		⋮
⋮	⋮		⋮				
?	?		?	B	B	B · · ·	B

(Profile 4)

By (**Not B) alternative B is not chosen, so by Fact 1 alternative K is still chosen. Finally raise B to the second position for voter r. Alternative K is still chosen or else r would not report this change. That is, K is chosen with the profile:

1	**2**	· · ·	**r − 1**	**r**	**r + 1**	· · ·	**n**
B	B	B · · ·	B	K	K	K · · ·	K
K	K	K · · ·	K	B			⋮
				⋮	⋮		⋮
⋮	⋮		⋮				
?	?		?		B	B · · ·	B

(Profile 5)

Now reconsider Profile 3 and suppose $G \neq K$ is chosen. Raise B to the top position for the first $r - 1$ voters, one at a time. By (**Not B) B is not chosen, so, by Fact 1, G is still chosen. Now raise B in voter r's profile to the second position:

1	**2**	· · ·	**r − 1**	**r**	**n + 1**	· · ·	**n**
B	B	B · · ·	B	K	?	· · ·	?
?	?		⋮	B			⋮
				⋮	⋮		⋮
⋮	⋮		⋮				
?	?		?		B	B · · ·	B

(Profile 6)

Suppose B is not selected. By Fact 1, alternative G is still chosen. From (*B) alternative B will be selected when it is raised to the top of r's profile. Hence, voter r should now falsely report this preference since she prefers B to $G \neq K$,contradicting strategy-proofness. Therefore, with Profile 6 alternative B is selected. Now raise K to the second position for voters 1 through $r-1$, and the first position for voters $r+1$ through n. Alternative B is still chosen or else the first group of voters would not truthfully report this change, whereas the second group would have misreported it. But this modified profile is the same as Profile 5, where K is chosen. A contradiction, so that $G \neq K$ cannot be chosen with Profile 3.

Step 3. Now consider an arbitrary profile in which r ranks some alternative $K \neq B$ on top. First modify the profile by dropping B to the bottom for all voters. From Step 2, alternative K is chosen. Now restore B to its initial position for all voters (one at a time). By Fact 1, either K or B is chosen.

Now consider:

1	**2**	· · ·	**r − 1**	**r**	**r + 1**	· · ·	**n**
B	B	$B \cdots$	B	B	A	A	A
⋮	⋮		⋮	⋮			⋮
					⋮		⋮
⋮	⋮		⋮	⋮			
C	C	C	C	C	C	$C \cdots$	C

(Profile 7)

where $C \neq B$ and $C \neq K$. Similarly to Step 1, have C jump in the rankings of the voters one at a time until we discover the pivotal voter m for alternative C. Symmetrically to Step 2, this pivotal voter has his top choice selected with Profile 7. On the other hand, from (*B) we also know that with Profile 7, alternative B is chosen. Hence, the pivot m must be such that $m \leq r$. But a symmetric argument (beginning with m and then finding r) shows that $r \leq m$, so $m = r$ and voter r is pivotal with respect to C as well as B. Thus, in addition to knowing that K or B is chosen for our initial arbitrary profile, we have that either K or C is chosen. Since $C \neq B$, alternative K is chosen.

Finally, if $K = B$ a similar argument shows that voter r is pivotal for A (as well as C) and that B is selected. Hence, r is a dictator.

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