Solutions

CS 441 Fall 2007 Exam 1 for the 1pm section (1:00-2:15pm)
There are 6 parts (A to F) on 9 pages with a total score of 90 points. Do all problems.
Calculators are not allowed.

Part A: Propositional and Predicate logics

1. (a) (5 points) Construct the truth table for $(\neg p \vee r) \wedge (\neg q \rightarrow r)$.

p	q	r	$\neg p$	$\neg p \vee r$	$\neg q$	$\neg q \rightarrow r$	$(\neg p \vee r) \wedge (\neg q \rightarrow r)$
T	T	T	F	T	F	T	T
T	T	F	F	F	F	T	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	F

(b) (1 point) Classify the above proposition. (circle one)

tautology contradiction contingency

Part A: (continue)

2. (12 points) Translate each English sentence (a-c) into logic and each logic proposition (d-f) into colloquial English. Let

   C(x): x has an email account. M(x,y): x has sent an email message to y.

   (a) James has sent an email message to Ken but Ken never replies.

   $M(\text{James}, \text{Ken}) \land \neg M(\text{Ken}, \text{James})$

   (b) Any person who doesn't have an email account never gets an email message.

   $\forall x \left[\neg C(x) \rightarrow \neg \exists y\, M(y, x) \right]$

   (c) There is a person who has emailed to everybody.

   $\exists x\, \forall y\, M(x, y)$

   (e) $\exists x\, \neg C(x)$

   Somebody doesn't have an email account.

   (g) $\exists x[C(x) \land \neg M(\text{Spammer},x)]$

   There is somebody with an email account who has never got email from Spam

   (h) $\forall x\, \exists y\, [C(x) \rightarrow M(y,x)]$

   Everybody who has an email account has got email from somebody.

Part B: Methods of Proof

1. (10 points) Prove that for any positive integer n, if n is divisible by 3, then $n(n+1)$ is divisible by 3. Note that if n is divisible by 3, then there exists an integer k such that $n = 3k$.
   Hint: Use one of the following strategies: Direct proof, Proof by contraposition, or Proof by contradiction.

Let n be any positive integer.

(We will show that if n is divisible by 3, then so is $n(n+1)$.)

Assume that n is divisible by 3.

(We will show that $n(n+1)$ is divisible by 3.)

Since n is divisible by 3, then $n = 3k$ where k is an integer.

$$n(n+1) = (3k)(3k+1)$$
$$= 3\left[k(3k+1)\right]$$

$k(3k+1)$ is an integer.

Since $n(n+1)$ can be written as 3 times an integer, then $n(n+1)$ is divisible by 3

Part C: Sets

1. (4 points) Suppose the universal set is U = the set of integers between 1 and 10 inclusive.
   Let A = { 1, 3, 5, 7 }
   Let B = { 1, 2, 3, 4 }
   Find the following sets.

   (a) $A \cup B$ $\{1, 2, 3, 4, 5, 7\} = \{1, 3, 5, 7, 2, 4\}$

   (b) $\bar{A} \cap B$ $\{2, 4\}$

2. (4 points) Suppose A = { a, b } and B = { 2, 3 }. Find the followings.

   (a) $B \times A$ $\{(2,a), (2,b), (3,a), (3,b)\}$

   (b) P(A) $\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

3. (10 points) True/False and short answers.

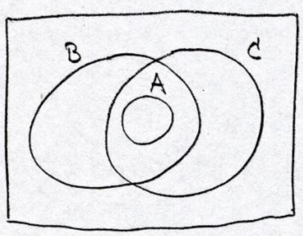
   (a) $\{ b, a \} \subseteq \{ a, \{\{ b, a \}\}, b \}$ **True** **False** (circle one)

   (b) $\emptyset \in \{ \{a\}, \{b\}, \{\emptyset\} \}$ **True** **False** (circle one)

   (c) Suppose $S = \{ a, \{a\}, \{a,\{a\}\}, \{a,a\}, \{\{a\},a\} \}$. Then $|S| =$ 3

   (d) If $A \subseteq B$ and $A \subseteq C$, then it is always the case that $A \subseteq B \cap C$. **True** **False** (circle one)

   (e) If $A \cap \bar{B} \neq \emptyset$, then $|A \cap B| = 0$ **True** **False** (circle one)



$\{a,a\} = \{a\}$

$\{a,\{a\}\} = \{\{a\},a\}$

$\therefore S = \{\underline{a}, \underline{\{a\}}, \underline{\{a,\{a\}\}}\}$

Part D: Functions

1. (6 points)
   Let A = { 6, 7, 8, 9, 10 }
   Let B = { a, b, c, d, e}
   Let f:A→B where f(6) = c, f(7) = b, f(8) = a, f(9) = c, and f(10) = e.

   (a) Determine f({6,8,10}).

   $\{c, a, e\}$

   (b) f is a one-to-one function. **True** **False** (circle one)

   (c) f is an onto function. **True** **False** (circle one)

2. (3 points)
   Let g:R→R where g(x) = 2x + 1
   Let h:R→R where h(x) = 3x + 5

   Determine (g ∘ h)(x).

$$\begin{aligned}(g \circ h)(x) &= g(h(x)) \\ &= g(3x+5) \\ &= 2(3x+5)+1 \\ &= 6x+11\end{aligned}$$

3. (3 points) Let f:R→R such that f(x) = 2⌈x/2⌉. What is the range of f?

   The set of all even numbers.

4. (2 points) Suppose g:A→B and |A| = |B| = 100. If g is not a 1-1 function, then g is not an onto function.

   True **False** (circle one)

Part E: Sequences and summation

1. (2 points) Write a formula for $5^2 + 6^2 + 7^2 + \ldots + 20^2$ using the summation symbol. Do not compute the value of the sum.

$$\sum_{k=5}^{20} k^2$$

2. (5 points) Compute the value of $\sum_{i=1}^{2}\sum_{j=i}^{i+2}(i+j)$.

$$= \sum_{j=1}^{3}(1+j) + \sum_{j=2}^{4}(2+j)$$

$$= \left[(1+1)+(1+2)+(1+3)\right] + \left[(2+2)+(2+3)+(2+4)\right]$$

$$= \left[2+3+4\right] + \left[4+5+6\right]$$

$$= 9 + 15$$

$$= 24$$

Part E: (continue)

3. (5 points) Compute the summation $\sum_{k=101}^{250}(3k+5)$. Note, after you replace all summations by appropriate formulas, you may leave the numbers unevaluated. For example, you may stop when your answer look like $8\cdot\frac{500^2\cdot 488}{6}-2000$. However, your answer should not look like $50^2+51^2+52^2+...+240^2$ because it contains ellipsis.

$$\sum_{k=101}^{250}(3k+5) = 3\sum_{k=101}^{250}k + \sum_{k=101}^{250}5$$

$$= 3\left[\sum_{k=1}^{250}k - \sum_{k=1}^{100}k\right] + (250-100)\cdot 5$$

$$= 3\left[\frac{250\cdot 251}{2} - \frac{100\cdot 101}{2}\right] + (250-100)\cdot 5$$

Part F: Mathematical Induction and Recursive Definition

1. (10 points) Use induction to prove that "for any positive integer n, $\sum_{k=1}^{n}(2k-1) = n^2$."

   This sentence is in the form $\forall n\, P(n)$.

   $P(n)$ is the statement $\sum_{k=1}^{n}(2k-1) = n^2$.

Basis step:

$P(1)$ is the statement $\sum_{k=1}^{1}(2k-1) = 1^2$. Since $\sum_{k=1}^{1}(2k-1) = (2\cdot 1-1) = 1$ and $1^2 = 1$, then $P(1)$ is true.

Induction step:

We will show that $\forall n \geq 1\, (P(n) \to P(n+1))$

Let n be any positive integer.

We will show that $P(n) \to P(n+1)$

Induction hypothesis: Assume $P(n)$

That is, $\sum_{k=1}^{n}(2k-1) = n^2$

We will show $P(n+1)$.

That is, $\sum_{k=1}^{n+1}(2k-1) = (n+1)^2$.

$$\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + 2(n+1)-1$$

$$= n^2 + 2n+1 \quad \text{from induction hypothesis}$$

$$= (n+1)^2$$

Thus, $P(n+1)$ is true.

Part F: (continue)

2. (4 points) Let f be a function defined below. Compute f(6).

   f(0) = 4
   f(1) = 10
   f(n) = | f(n-1) − f(n-2) | - 1 for $n \geq 2$

n	0	1	2	3	4	5	6
$f(n-1)-f(n-2)$	–	–	6	-5	-1	-4	3
$f(n)$	4	10	5	4	0	3	2

$\rightarrow f(6) = 2$

3. (4 points) Let S be a set defined below. Circle all (and only) elements of S that are smaller than or equal to 20.

   1 ∈ S.
   If x ∈ S, then x+4 ∈ S and 3x ∈ S.
   Nothing else is in S.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

$f(0) = 4$

$f(1) = 10$

$f(n) = |f(n-1) + f(n-2)| - 1$ for $n \geq 2$

n	0	1	2	3	4	5	6
$f(n)$	4	10	13	22	34	55	88

$\rightarrow f(6) = 88$